Options Payoffs Perspective on Financial Engineering

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ABSTRACT

The option theory and its applications play an important role in modern finance. Many trading strategies, corporate incentive plans, and hedging strategies include various types of options. In this paper, the option definitions and basics of how the instruments are traded will not be the main focus but in particular, we will focus on the critical aspects of the dynamics of payoff and profit/loss functions, the difference between European versus American options, the Binomial option model with/without dividends, and different trading strategies.

INTRODUCTION

Option Structure Details and Exercise types

The key difference between a European and American option is that a European option can only be exercised on the maturity date, while an American option can be exercised any time before or on its maturity date. Given the extra flexibility of the American option, its price (option premium) should be higher than or equal to its European counterpart.

Call Option Premium$_{\text{European}}$ <= Call Option Premium$_{\text{American}}$

Put Option Premium$_{\text{European}}$<= Put Option Premium$_{\text{American}}$

In quantitative finance, another thing concerns us. For a European option, we have a closed-form solution; that is, the Black-Scholes-Merton option model. However, we don't have a closed-form solution for an American option. To price an American option, we will have to use the slightly more complex and computationally intensive Binomial Tree Method (also called the CRR method).

Cash Flows, Types of Options, a Right, and an Obligation

We know that, for each business contract, we have two sides: a buyer and a seller. This is also true for an option contract. A call buyer will pay upfront (cash output) to acquire a right. Since this is a zero-sum game, a call option seller will enjoy an upfront cash inflow and assume an obligation. The following table presents these positions (buyer or seller), directions of the initial cash flows (inflow or outflow), the option buyer's rights (buy or sell), and the option seller's obligations (that is, to satisfy the option seller's demand):

Buyer Seller European

A **call option**, often simply labeled a "**call**", is a financial contract between two parties, the buyer and the seller of this type of option. The buyer of the call option has the right, but not the obligation, to purchase/buy an agreed quantity of a particular commodity/asset or financial instrument (the underlying asset) from the seller for a certain given price (the strike/exercise price) on or before a given date(expiration date).

The seller (or "writer") is obligated to sell the commodity or financial instrument to the buyer if the buyer so decides. The buyer pays a fee (called a premium) for this right. The term "call" comes from the fact that the owner has the right to "call the stock away" from the seller.

Put: on the other hand gives owner the right to sell an asset for a given price on or before the expiration date.

Properties of Options

For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.

Key elements in defining an option**:**

S: Price of stock/Underlying asset now
S_T: Exercise price (strike price) at T
B: Price of discount bond with face value $1 and maturity T (clearly, $B \leq 1$)
C: Price of a European call with strike price and maturity T (today is 0)
P: Price of a European put with strike price K and maturity T
c: Price of an American call with strike price K and maturity T
p: Price of an American put with strike price K and maturity T .

Option Value and Asset Volatility Option value increases with the volatility of underlying asset.

Option Pricing

Option pricing refers to the amount per share at which an option is traded. Options are derivative contracts that give the holder (the "buyer") the right, but not the obligation, to buy or sell the underlying instrument at an agreed-upon price on or before a specified future date. Although the holder of the option is not obligated to exercise the option, the option writer (the "seller") has an obligation to buy or sell the underlying instrument if the option is exercised.

Depending on the strategy, options trading can provide a variety of benefits, including the security of limited risk and the advantage of leverage. Another benefit is that options can protect or enhance your portfolio in rising, falling and neutral markets. Regardless of why you trade options – or the strategy you use – it's important to understand how options are priced. In this tutorial, we'll take a look at various factors that influence options pricing, as well as several popular options-pricing models that are used to determine the theoretical value of options.

Option Payoff

The payoff of an option on the expiration date is determined by the price of the underlying asset.
Example: Consider a European call option on IBM with exercise price $100. This gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at $100 on the expiration date. Depending on the share price of IBM on the expiration date, the option owner's payoff looks as follows:

IBM Price	Action	Payoff
.	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
.	Exercise	S_T-100

Note:
• The payoff of an option is never negative.
• Sometimes, it is positive.
• Actual payoff depends on the price of the underlying asset.
Payoffs of calls and puts can be described by plotting their payoffs at expiration as function of the price of the underlying asset:
The net payoff from an option must include its cost.

Call Option Payoff Formula
The total profit or loss from a long call trade is always a sum of two things:
Initial cash flow
Cash flow at expiration.

Initial cash flow
Initial cash flow is constant – the same under all scenarios. It is a product of three things:
The option's price when you bought it
Number of option contracts you have bought
Number of shares per contract
Usually you also include transaction costs (such as broker commissions).
If initial option price (including commissions) is $2.88 per share, we are long 1 contract of 100

shares, therefore initial cash flow is:
2.88 x 1 x 100 = – \$288
Of course, with a long call position the initial cash flow is negative, as you are buying the options in the beginning.

Cash flow at expiration
The second component of a call option payoff, cash flow at expiration, varies depending on underlying price. That said, it is actually quite simple and you can construct it from the scenarios discussed above.
If underlying price is below than or equal to strike price, the cash flow at expiration is always zero, as you just let the option expire and do nothing.
If underlying price is above the strike price, you exercise the option and you can immediately sell it on the market at the current underlying price. Therefore the cash flow is the difference between underlying price and strike price, times number of shares.
CF = what you sell the underlying for – what you buy the underlying for when exercising the option
CF per share = underlying price – strikes price
CF = (underlying price – strike price) x number of option contracts x contract multiplier
In our example with underlying price 49.00:
CF = (49 – 45) x 1 x 100 = \$400
Putting all the scenarios together, we can say that the cash flow at expiration is equal to the greater of:
(underlying price – strike price) x number of option contracts x contract multiplier
Zero

Call B/E = strike price + initial option price
In our example with strike = 45 and initial price = 2.88 the break-even point is 47.88. You can try to use this as underlying price in the P/L formula above and you will get exactly zero profit.
Long Call Option Payoff Summary
A long call option position is bullish, with limited risk and unlimited upside.
Maximum possible loss is equal to initial cost of the option and applies for underlying price below than or equal to the strike price.
With underlying price above the strike, the payoff rises in proportion with underlying price.
The position turns profitable at break-even underlying price equal to the sum of strike price and initial option price.

METHODOLOGY
Binomial Option Pricing Model
Determinants of Option Value
Key factors in determining option value:
1. Price of underlying asset S
2. strike price K
3. time to maturity T
4. interest rate r

5. dividends D
6. Volatility of underlying asset s.

Additional factors that can sometimes influence option value:

7. Expected return on the underlying asset
8. Additional properties of stock price movements
9. Investors' attitude toward risk,...

Price Process of Underlying Asset

In order to have a complete option pricing model, we need to make additional assumptions about

1. Price process of the underlying asset (stock)
2. Other factors.

We will assume, in particular, that:

• Prices do not allow arbitrage.
• Prices are "reasonable".
• A benchmark model — Price follows a binomial process.

S_{up}

S_0

S_{down}

t=0 t=1 time

One-period Binomial Model

Example: Valuation of a European call on a stock.

• Current stock price is \$50.
• There is one period to go.
• Stock price will either go up to \$75 or go down to \$25.
• There are no cash dividends.
• The strike price is \$50.
• one period borrowing and lending rate is 10%.

The stock and bond present two investment opportunities:

75 1.1

50 1

25 1.1

The option's payoff at expiration is:

25

C_O

0

Example: What is C_O, the value of the option today?

Claim: We can form a portfolio of stock and bond that gives identical payoffs as the call.
Consider a portfolio (a, b) where
- *a* is the number of shares of the stock held
- *b* is the dollar amount invested in the riskless bond.

We want to find (a, b) so that
75a +1.1b =25
25a +1.1b =0.
There is a unique solution
a =0.5 and b = -11.36.
That is buy half a share of stock and sell $11.36 worth of bond payoff of this portfolio is identical to that of the call present value of the call must equal the current cost of this "replicating portfolio" which is
(50)(0.5)-11.36=13.64.

Definition: Number of shares needed to replicate one call option is called hedge ratio or option delta.
In the above problem, the option delta is a:
Option delta =1/2.

RESULTS

Payoff and Profit/Loss Functions for Call Options
As we know, an option gives its buyer the right to buy (call option) or sell (put option) something in the future to the option seller at a predetermined price (exercise price). For example, if we buy a European call option to acquire a stock
for X dollars, such as $45, at the end of three months, our payoff on maturity day
will be calculated using the following formula:

Call Option Payoff= Max(St-X,0)

Here, S_t is the stock price at the maturity date, (T), and X represents the strike price or the exercise price (45$). Assume the stock price is $30 three months later. We will not exercise our call option to pay $45 in exchange for the stock since we could buy the same stock at $30 on the open market. On the other hand if the stock price is $60, we will most definitely be keen on exercising this option as it will give us a profit of $15 per contract.
Let us now code a program using Python programming Language to graphically represent the generic payoff function call options assuming a few representative values for stock price and strike price.

```
import numpy as np
import matplotlib.pyplot as plt

if __name__ == '__main__':

def payoffFuncCall(sT,x):
```

```
return (sT-x+abs(sT-x))/2
s = np.arange(5,200,3)
x=30
# Figure setup
fig=plt.figure()
axis=fig.add_subplot(111)
payoff=payoffFuncCall(s,x)
# Set up axis details
axis.set_ylim(-10,200)
axis.set_xlabel('Stock Price at Maturity')
axis.set_ylabel('Payoff at Maturity')
axis.grid(True)
plt.plot(s,payoff,color='orangered',label='Call Option
Payoff',linewidth=3)
plt.title("Payoff function for Call Options")
plt.show()
```

Check out the contour of the output graph. The flat horizontal line followed by an upward slope is a common symbology of call options.

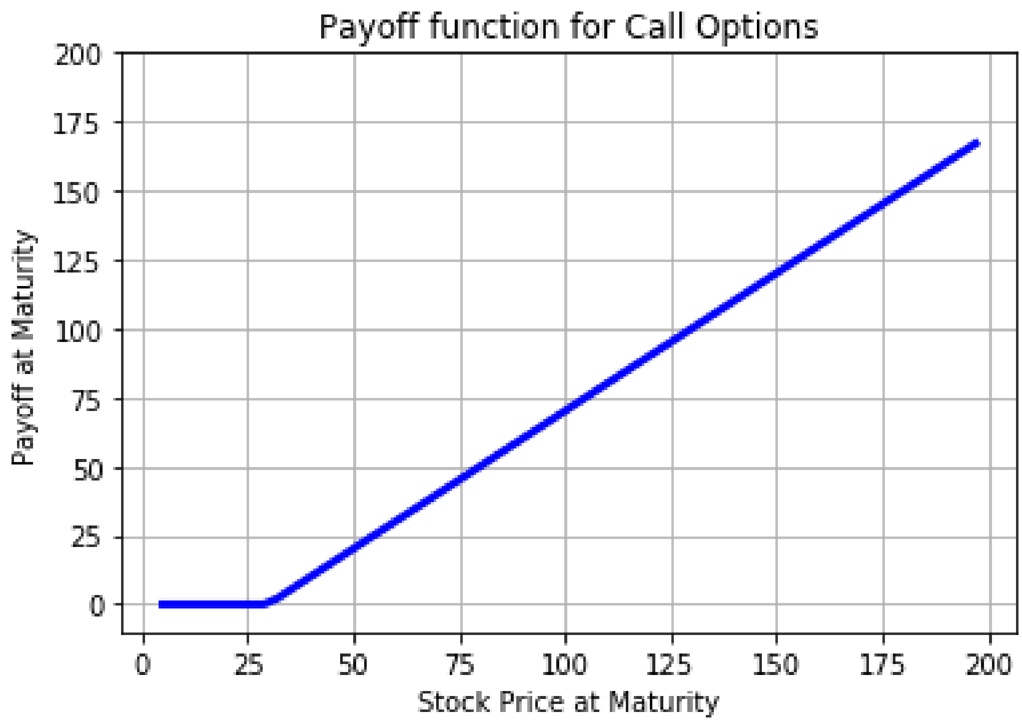

Figure 1: Payoff and Profit/Loss Functions for Call Options

Call Option—Buyer's Payoff Vs. Sellers Payoff

The payoff for a call option seller is the opposite of that of the buyer. It is important to remember that option buying and selling is a zero-sum game: When one party makes a profit, it invariably means that the other has lost money.

For example, assume an institutional house sold three call options with an exercise price of $30. When the stock price is $35 on maturity, the option buyer's payoff is $15, while the total loss to the option writer is also $15. If we represent the call option premium by “c,” the profit/loss function for a call option buyer is the difference between the option price on exercise date (this will closely mirror the underlying asset price) and the initial premium paid for holding the option.

Here, we ignore the time value of money since maturities are usually quite short.
For a call option buyer, the profit is calculated using the following formula:

For a call option seller, the profit is calculated using the following formula:
Buyer’s Payoff for Call Option= $Max(S_t-X,0)-c$

call option buyer and seller:

Sellers Payoff for Call Option= $c-Max(S_t-X,0)$

Check out the following code for a graph showing the profit/loss functions for the

```
import numpy as np
import matplotlib.pyplot as plt

if __name__ == '__main__':

    s = np.arange(10,100,5)
    x=57;
    call=3.2
    profitCalc=(abs(s-x)+s-x)/2 -call
    y2=np.zeros(len(s))
    # Figure setup
    fig=plt.figure()
    axis=fig.add_subplot(111)
     # Set up axis details
    axis.set_ylim(-30,50)
    plt.plot(s,profitCalc,label='Call Option Buyer\'s
payoff',color='teal',linewidth=3)
    plt.plot(s,y2,'-.')
    plt.plot(s,-profitCalc,label='Call Option Seller\'s
payoff',color='deeppink',linewidth=3)
    plt.title("Profit/Loss function")
    axis.set_xlabel('Stock price at Maturity')
    axis.set_ylabel('Profit (loss) at Maturity')
```

```
axis.grid(True)
plt.legend()
plt.show()
```

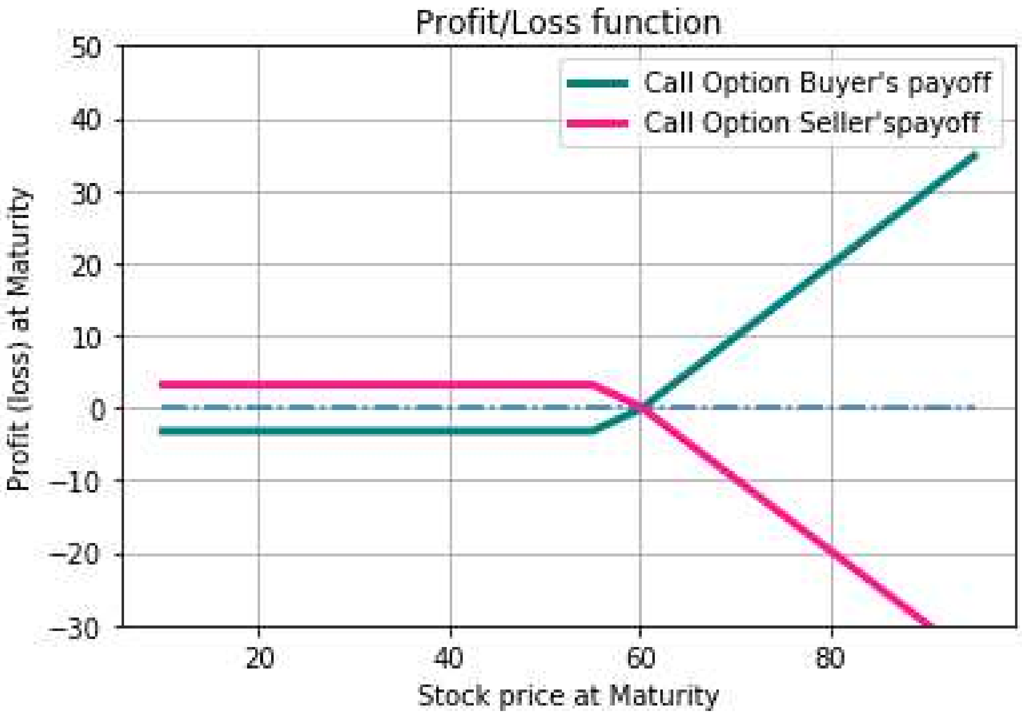

Figure 2: Call Option-Buyer's Payoff Vs. Sellers Payoff

Put Option—Buyer's Payoff Vs. Sellers Payoff

A put option gives its buyer the right to sell a security (commodity) to the put option buyer in the future at a predetermined price, X. The following is its payoff function:

Put Option Payoff= Max(X-S_t,0)

Here, S_t is the stock price at the maturity date, (T), and X represents the strike price or the exercise price. Consider the fact that the initial premium paid for a put option is p. Then, for a put option buyer, the profit/loss function is as follows:

Buyer's Payoff for Put Option= Max(X-St,0)-p

For a put option seller, the profit is calculated using the following formula:

Sellers Payoff for Put Option= p- Max(X-St,0)

Check out the following code for a graph showing the profit/loss functions for the put option buyer and seller.

```
import numpy as np
```

```
import matplotlib.pyplot as plt
if __name__ == '__main__':
    s = np.arange(10,100,5)
    x=57;
    put=3.2
    profitCalc=put-(abs(x-s)+x-s)/2
    y2=np.zeros(len(s))
    x3=[x, x]
    y3=[-30,10]
    # Figure setup
    fig=plt.figure()
    axis=fig.add_subplot(111)
    # Set up axis details
    axis.set_ylim(-30,50)
    plt.plot(s,profitCalc,label='Put Option Seller\'s
payoff',color='firebrick',linewidth=3)
    plt.plot(s,y2,'-.')
    plt.plot(s,-profitCalc,label='Put Option Buyer\'s
payoff',color='limegreen',linewidth=3)
    plt.plot(x3,y3,label='Exercise Price',color='gold',linewidth=5)
    plt.title("Profit/Loss function")
    axis.set_xlabel('Stock price at Maturity')
    axis.set_ylabel('Profit (loss) at Maturity')
    axis.grid(True)
    plt.legend()
    plt.show()
```

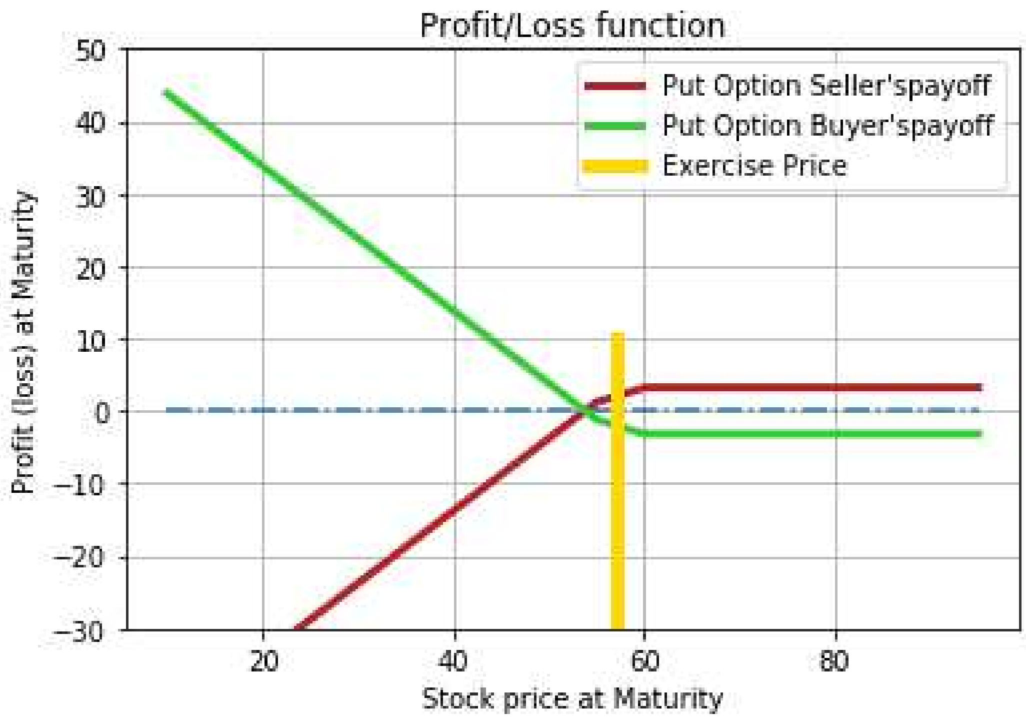

Figure 3**: Put Option-Buyer's Payoff Vs. Seller's Payoff**

CONCLUSIONS

In summary, profit and loss diagrams have proven useful for a visual representation regarding the gain and loss potential for various option strategies.

Furthermore, when using these diagrams, we get a better understanding of how to replicate the gain/loss profile of different investment strategies using options. This can be useful to further understand and visualize many investment strategies.

These simple graphs represent the profit and loss potential, at expiration, assuming the positions are closed at their intrinsic value, if any. As with any investment strategy, you should consider and understand all of the risks associated with that particular strategy and how it fits into your objectives, risk profile and portfolio prior to making an investment decision.

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On Change Point Detection in A Series of Stimulus-Response Data

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Abstract
In this paper, we demonstrate the power of functional data models for a statistical analysis of stimulus-response experiments which is a quite natural way to look at this kind of data and which makes use of the full information available. In particular, we focus on the detection of a change in the mean of the response in a series of stimulus-response curves where we also take into account dependence in time.

Keywords*:* stimulus-response data, functional data, functional time series, changepoint test, inhibitory synaptic transmission

1. Introduction

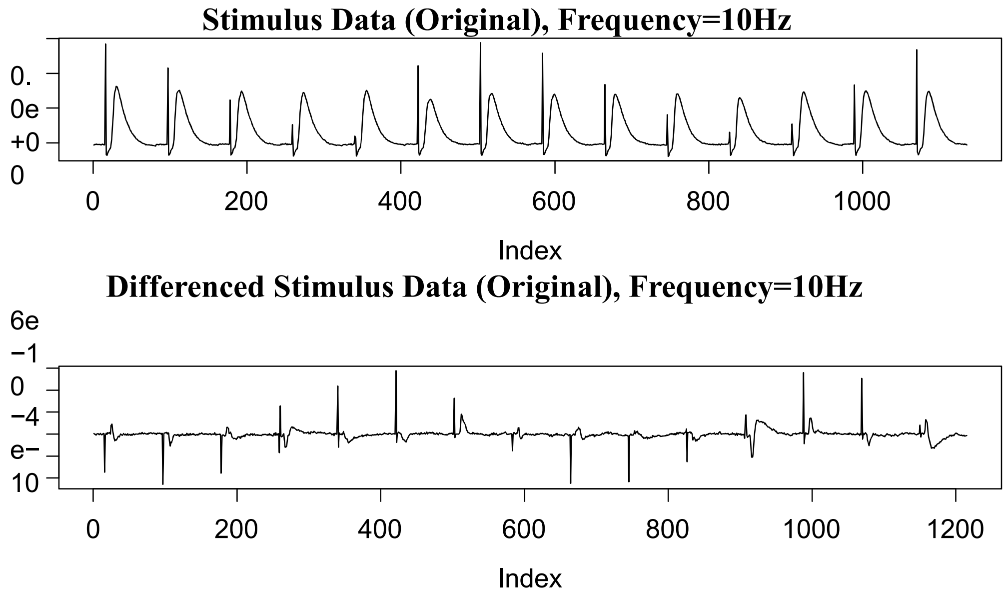
Stimulus-response data are a frequent product of cognitive experiments. The test object is confronted with a stimulus, and the following response is measured in some form, e.g. as the changes in time of the potential at certain locations in a single neuron or by means of an electroencephalogram (EEG) of an animal or human. The full data are functions of time or, in the EEG case, vectors of functions. Usually, they are already digitized during storage, but with such a fine discretization such that they still can be seen as continuouscurves.

For a statistical analysis, we have to model such data as random functions of time. However, in cognitive science, the full information available is rarely used for inference. Usually, the response curves are reduced to a low-dimensional data vector before, e.g., performing statistical tests. Those data vectors consist of simple univariate characteristics like maximal response, average response, length of response, response latency, i.e. waiting time between stimulus and response etc. Modern functional data analysis allows to use the full information of the response curves in a quite natural manner which we want todemonstrate in this paper with a real-data example.

Standardizing the observation interval to [0,1], let $X_i(t), 0 \leq t \leq 1,$ denote the response curve from the i^{th}experiment. Analogously to multivariate data, the mean curve of functional data is defined pointwise, i.e. $\mathrm{E}X_i(t) = \mu(t), 0 \leq t \leq 1,$ if the functional data X_i have identical means. As for random vectors, there are tests for equality of the mean to some given function in case of one sample or for equality of the means of two independent samples (compare, e.g., Horva´th and Kokoszka (2010), chapter 5). In this paper, we consider a more involved testing problem. We have a time series of response curves $X_i(t), 0 \leq t \leq 1, i = 1,...,N,$ generated by presenting the same stimulus repeatedlyto the same test object. The particular kind of data are explained in chapter 2.

In section 3, we consider the problem of testing for a change in the mean under the assumption of independent $X_1,...,X_N$as well as in the general setting of dependent curves. Such changepoints are of interest in experiments about learning or increasing fatigue of the test object under repeated stimuli. E.g., the response latency may become longer corresponding to a shift of the response curve towards the time of stimulus, or the response curves may become flatter on the average corresponding to the test object getting used to the stimulus. In chapter 4, we finally apply the methods described in chapter 3 to our actual stimulus response data and detect various changes in the mean in our sequences of stimulus-response curves. We also test the detrended data for dependence. It turns out that subsequent curves are dependent which has to be taken into account in the tests forchanges in the mean.

Figure 1: Original stimulus-response data



2. Preprocessing the data

The data are generated by stimulus-response experiments on a single on a single neuron in the lateral superior olive, as part of a larger research project on the reliability of inhibitory synaptic transmission in the auditory brainstem. For more details about the physiological background, we refer to Fischer (2016) or Kraechan et al. (2016). The stimulus is a brief electric shock that triggers synaptic activity and is repeatedly applied at various frequencies (1, 2, 5, 10, 50 Hz). The duration of the experiment is always 1 min such that the sample sizes for the samples with different stimulus frequencies vary between $N = 60$ for 1 Hz and $N = 3000$ for 50 Hz. The individual responses are short-lived enough such that each response has ended well before the next stimulus even in case of the highest stimulus frequency. Hence, we have a series of curve data which look similar, but show some randomvariation.

The top panel of Figure 1 shows a subsection of 14 curve data from the experiment with stimulus frequency 10 Hz (observations number 11-24), where the total sample size was $N = 600$. Note that the horizontal axis shows the index number of discretized single measurements recorded for storage, not some physical time. We always stored about 85observations for each individual stimulus-response cycle independently of the frequency.

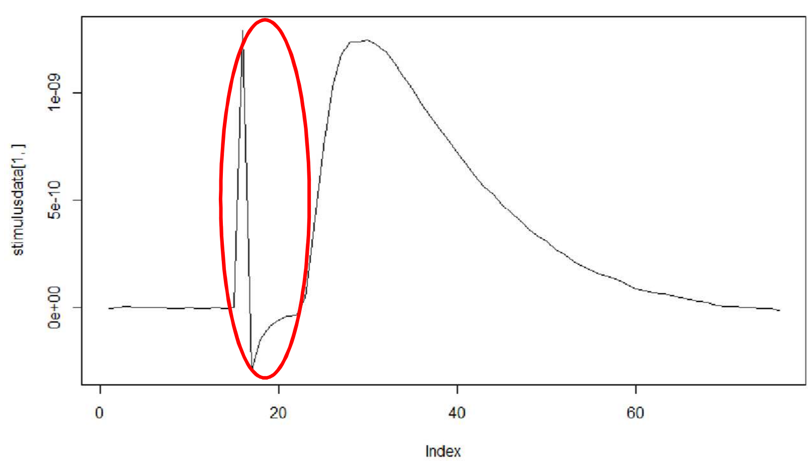
For the mean tests, we use the response curves themselves. In testing for dependence of the data, it is however convenient to first apply a differencing filter which removes the mean even in situations where it is slowly changing. To be precise, if $X_i(t)$ denote theoriginal response curves, then the differenced curve data are the random functions

$$Y_i(t) = X_{i+1}(t) - X_i(t), \quad i = 1,\ldots,N-1. \tag{1}$$

The lower panel of Figure 1 shows a subsection of the differenced response curves from the experiment with stimulus frequency 10 Hz.

At the beginning of each response and differenced response there is a noticeable sharp spike (circled in red) in Figure 2. This is an artifact which represents the direct effect of the stimulus onto the measuring device, but does not correspond to the response of the cell. The cell reacts to the stimulus only after a brief delay. As the stimulus part and the response part of the curves are well enough separated and we are only interested in the measurements of the response, it is safe to remove a few data points at the beginning of each curve. We therefore cut the data points in the circle and consider only the rest as the response curve to be analyzed further on. Once the truncation has been done, we have 68, 73, 78, 73, 73 measurement points per individual curve left in the case of 1, 2, 5, 10 and 50 Hz frequencies respectively, which are then smoothed to form the curves shown in thefigures.

Figure 2: Artifact



Figures 3, 4 and 5 show the adjusted and differenced plots of parts of the response curve samples corresponding to stimulus frequencies 1, 2, 5, 10 and 50 Hz respectively. In particular, after the adjustment the local random noise in the differenced data can be seen much more clearly.

Figure 3: Adjusted Responses (left) and their Differenced Counterparts 1, 2 Hz

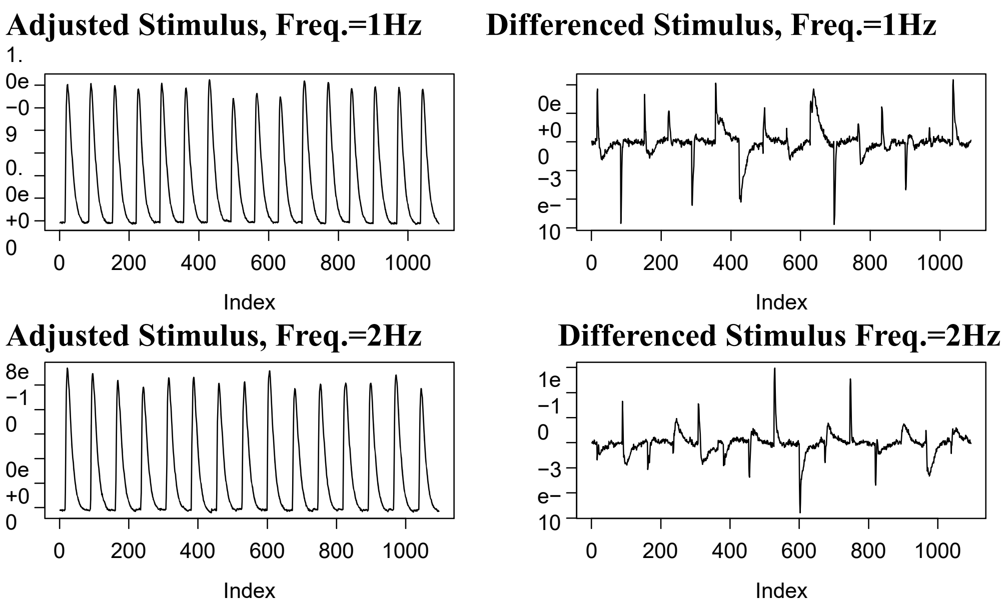


Figure 4: Adjusted responses (left) and their differenced counterparts 5, 10 Hz

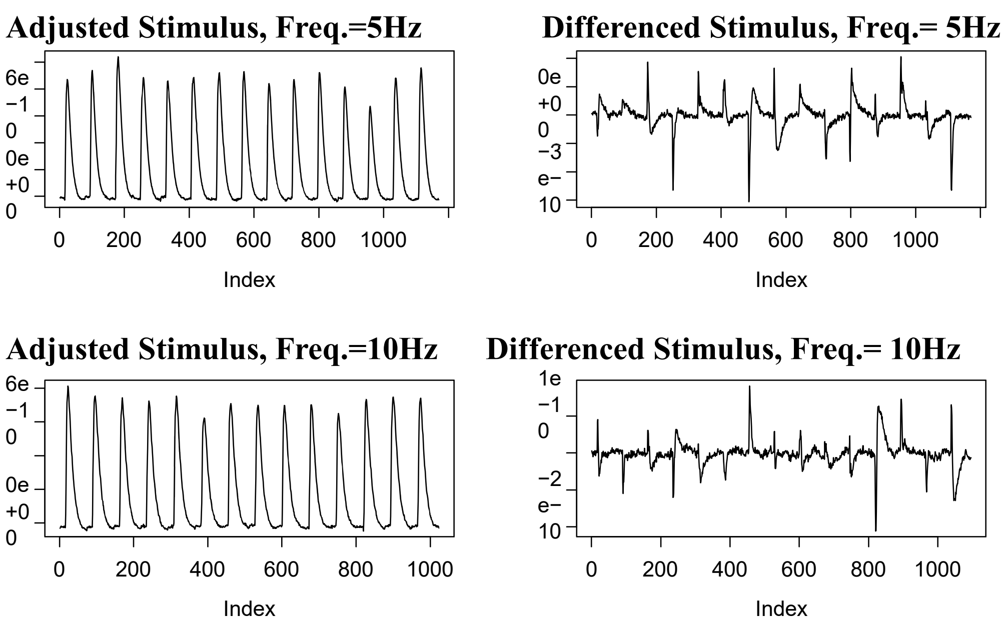
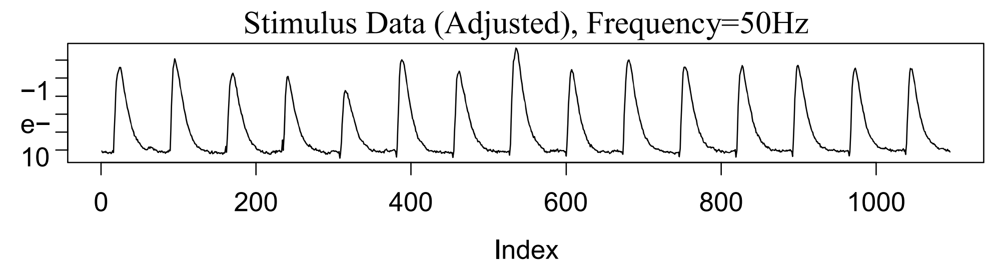
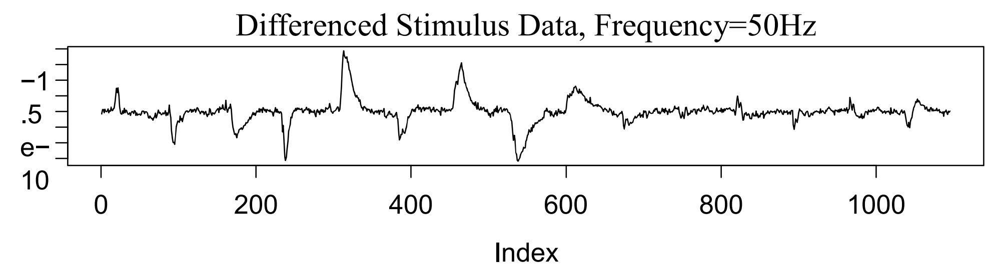


Figure 5: Adjusted responses (top) and their differenced counterparts 50 Hz





3. Testing for changes in the mean

We interpret observed response curves resp. their transformations after preprocessing asrandom functions $X_i(t), 0 \le t \le 1$, and we assume that they are square integrable:

$$\int_0^1 X_i^2(t)dt < \infty ,$$

i.e. X_i is a random variable with values in the space H = $L^2[0,1]$ of, for convenience complex-valued, square integrable functions on [0,1]. This space is a separable Hilbert space which has a quite similar structure as the finite dimensional Euclidean space R^m. Inparticular, there is a scalar product and a corresponding norm

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\ dt, \quad \|f\| = \left\{\int_0^1 |f(t)|^2 dt\right\}^\wedge 1/2\ , \quad f, g \in \boldsymbol{H}$$

where$\overline{g(t)}$denotes the complex conjugate of $g(t)$. There exists a countable orthonormal basis, i.e. a sequence of functions $v_1, v_2, \ldots$in H with $\|v_k\| = 1,\ \langle v_k, v_l\rangle = 0$ for all $k \neq l$, such that we have the usual linear expansion of any f in H in terms of the basis

$$f(t) = \sum_{k=1}^{\infty} \langle f, v_k\rangle v_k(t) \qquad \|f\|^2 = \sum_{k=1}^{\infty} \langle f, v_k\rangle^2$$

If we choose, in particular, the Fourier basis $v_k(t) = e^{i2\pi kt} = \cos(2\pi kt) + i\sin(2\pi kt), -\infty < k < \infty$, then this is the Fourier expansion well known in signal analysis, and $\langle f, v_k\rangle$ are the Fourier coefficients of f. In the following, we refer some notions and results from chapter 6 of Horváth and Kokoszka (2010).

3.1 Changepoint test for independent data

If $X_i(t), i = 1,\ldots,N$, is a sequence of real-valued random functions in H, then we decompose them into the mean function and the random component:

$$X_i(t) = \mu_i(t) + Y_i(t), \qquad \mathrm{E}Y_i(t) = 0.$$

We assume that the random components Y_i are independent and all have the same distribution satisfying

$$E\|Y_i\|^2 = \int_0^1 Y_i^2(t)dt < \infty$$

Then, the covariance function measuring dependence between the function values $X_i(t), X_i(s)$ at different points t,s in time, does not depend on i:

$$c(t,s) = \mathrm{cov}(X_i(t), X_i(s)) = \mathrm{E}Y_i(t)Y_i(s) \quad \text{for all } i, 0 \le s,t \le 1,$$

and it allows for the expansion

$$c(t,s) = \sum_{k=1}^{\infty} \lambda_k v_k(t) v_k(s).$$

$\lambda_1 \geq \lambda_2 \geq \ldots$ are the ordered eigenvalues, which automatically are nonnegative, and $v_1, v_2, \ldots$ the corresponding orthonormal eigenfunctions of the covariance operator C which linearly maps a function f in H onto the function Cf given by

$$(Cf)(t) = \mathrm{E}\big(\langle Y_i, f\rangle Y_i(t)\big) = \mathrm{E}\Big(\int_0^1 Y_i(s)f(s)dsY_i(t)\Big) = \int_0^1 c(t,s)f(s)ds. \tag{2}$$

The functions $v_1, v_2, \ldots$ are called the functional principal components. As they are an orthonormal basis of H, we also have

$$Y_i(t) = \sum_{k=1}^{\infty} y_{i,k} v_k, \text{ where } y_{i,k} = \langle Y_i, v_k\rangle$$

We want to test if the response curves are on the average identical or, if at some unknown changepoint m in the sample, the mean changes. In our model above, the null hypothesis H_0 of no change and the alternative H_1 of one change are

$$H_0\colon \mu_1 = \cdots = \ H_1\colon \mu_1 = \cdots = \ \mu_m \neq \mu_{m+1} = \cdots = \mu_N \text{ for some } 1 \le m < N$$

As the basis of the test statistic, we consider the partial means of data before and after k:

$$\hat{\mu}_k(t) = \frac{1}{k}\sum_{i=1}^{k} X_i(t) \quad \tilde{\mu}_k(t) = \frac{1}{N-k}\sum_{i=k+1}^{N} X_i(t)$$

Under H_0, both$\hat{\mu}_k$ and $\tilde{\mu}_k$will estimate the common mean of all the functional data and will be approximately equal for all k. If, however, there is a changepoint$m < N$, then $\hat{\mu}_k - \tilde{\mu}_k$will be large for $k \approx m$.

For small k, the variability of $\hat{\mu}_k$ is rather large, as only few observations contribute to the average, and the same applies to $\tilde{\mu}_k$for small $N - k$. Therefore, the test uses the weighted differences

$$P_k(t) = \frac{k(N-k)}{N}\left(\hat{\mu}_k(t) - \tilde{\mu}_k(t)\right) = \sum_{i=1}^{k} X_i(t) - \frac{k}{N}\sum_{i=1}^{N} X_i(t),$$

to take into account the different random variability of $\hat{\mu}_k - \tilde{\mu}_k$for various k.
If $P_k$would be scalar numbers, we would look at the maximum value of $|P_k|$in thespirit of classical changepoint analysis and reject the hypothesis H_0 if it exceeds a critical bound depending on the level of the test. However, $P_k$is a function in H. We could reduce them to scalar characteristics like the integral of the absolute value or the maximum if we would have a rather precise notion about the type of change to expect. A main feature offunctional data analysis, however, is its flexibility regarding the characterization of response curves. So, we are looking for several scalar quantities which combined give us the essential features of the whole function. For a suitable d (compare subsection 4), these are just the scores of $P_k$relative to the first d functional principal components $v_1,...,v_d$, i.e.

$$\langle P_k, v_\ell\rangle = \int_0^1 P_k(t)v_\ell(t)dt, \ \ell = 1, \ldots, d.$$

Then, for convenience, we look at a suitable weighted average of the squares, not of the absolute values, of the $\langle P_k, v_l\rangle$:

$$T_N(k) = \frac{1}{N}\sum_{\ell=1}^{d}\frac{1}{\lambda_\ell}\langle P_k, v_\ell\rangle^2$$

This is not yet a feasible test statistic, as it depends on the unknown v_l, λ_l. First notethat

$$\langle P_k, v_\ell\rangle = \langle\sum_{i=1}^{k} X_i - \frac{k}{N}\sum_{i=1}^{N} X_i, v_\ell\rangle = \langle\sum_{i=1}^{k} Y_i - \frac{k}{N}\sum_{i=1}^{N} Y_i, v_\ell\rangle = \sum_{i=1}^{k} y_{i,\ell} - \frac{k}{N}\sum_{i=1}^{N} y_{i,\ell}$$

as centering each summand in both sums by subtracting X_N has no effect. Therefore, for estimating

$T_N(k)$, we need to estimate $\lambda_l, y_{i,l}, l = 1, \dots, d, i = 1, \dots, N$. First we estimate the covariance function $c(t,s)$ by the sample version

$$\hat{c}(t,s) = \frac{1}{N}\sum_{i=1}^{N}\big(X_i(t) - \overline{X}_N(t)\big)\big(X_i(s) - \overline{X}_N(s)\big)$$

where, under the hypothesis of no change, the sample mean $\bar{X}_N(t)$of $X_1(t),\dots,X_N(t)$ estimates the common mean function of the curve data. $\hat{c}(t,s)$characterizes the estimate $\hat{C}$of the covariance operator analogously to (2). Finally, we have to calculate the first d eigenvalues $\hat{\lambda}_1 > \dots > \hat{\lambda}_d$ and the scalar products of the centered data with the corresponding eigenvectors $\hat{v}_1, \dots, \hat{v}_d$ of $\hat{C}$to get the estimate of $T_N(k)$

$$\widehat{T}_N(k) = \frac{1}{N}\sum_{\ell=1}^{d}\frac{1}{\hat{\lambda}_\ell}\left(\sum_{i=1}^{k}\hat{y}_{i,\ell} - \frac{k}{N}\sum_{i=1}^{N}\hat{y}_{i,\ell}\right)^2$$

These calculations can be easily done using the R package fda. There are various possibilities how to combine$\widehat{T}_N(k), k = 1, \dots, N$ to a single scalar test statistic. Horváth andKokoszka (2010) just use averaging and get

$$S_{N,d} = \frac{1}{N}\sum_{k=1}^{N}\widehat{T}_N(k) = \frac{1}{N^2}\sum_{\ell=1}^{d}\frac{1}{\hat{\lambda}_\ell}\sum_{k=1}^{N}\left(\sum_{i=1}^{k}\hat{y}_{i,\ell} - \frac{k}{N}\sum_{i=1}^{N}\hat{y}_{i,\ell}\right)^2$$

$H_0$is rejected if $S_{N,d}$is large. Let us just summarize again the intuition behind this decision procedure. As mentioned above, if the mean does not change, the weighted differences $P_k(t)$ of the sample mean functions before and after k should all be reasonably close to 0. Hence, for all k and l, their squared scores $\langle P_k(t), v_l\rangle^2$ should be small. Now, $T_N(k)$ as a weighted average of those quantities should be small too for $k = 1,\dots,N$, and, hence, this also holds for the average over k. If we replace the unknown quantities in this average by their sample analogues, then we just get $S_{N,d}$.

Finally, we need critical values for the test which we get from the asymptotic distribution of $S_{N,d}$under the hypothesis which has been derived by Horváth and Kokoszka (2010) undersome rather weak regularity assumptions. In particular, for $N \to \infty$

$$pr\big(S_{N,d} > z \mid H_0 \; holds\big) \to \; K_d = \int_0^1 \sum_{l=1}^{d} B_l^2(t)dt, \qquad (3)$$

where$B_l, l = 1, \dots, d,$ are independent standard Brownian bridges. The distribution of $K_d$has been derived quite early by Kiefer (1959) in his study of extensions of the Cramér-von Mises test. Critical values for $S_{N,d}$for various significance levels and values of d can be found in Table 6.1 of Horváth and Kokoszka (2010).

If the test rejects the hypothesis and detects a changepoint$m$, then we are interested in estimating its location. A consistent estimate $\widehat{m}$is derived by checking at which index$k$, the statistic$\widehat{T}_N(k)$ assumes its maximum:

$$\widehat{T}_N(\hat{m}) = \max_{k=1,\ldots,N} \widehat{T}_N(k) \tag{4}$$

Note that if we detect a changepoint, we can say that the mean is not constant over time, i.e. H_0 does not hold, up to the usual small error probability. It does not necessarily imply that the mean is constant before and after the changepoint. The test is also sensitive against other kinds of alternatives, e.g. several changepoints or a gradual change of themean.

One way to check the constancy of the mean before and after the detected changepoint is a repeated application of the test. So, if H_0 is detected and $\widehat{m}$ is the estimated changepoint, we apply the test again twice to the samples $X_1, \ldots, X_{\widehat{m}}$resp.$X_{\widehat{m}+1}, \ldots, X_N$. If we detect some changepoints in those subsamples, then again we split the samples and apply the test again until finally we get a partition of the original data into subsamples which all have approximately constant means or just have small enough sample sizes that the test does not reject the hypothesis any longer.

3.2 Changepoint test for dependent data

We now consider the same setting as in the previous subsection, but we allow for dependence of the curve data. In particular, we assume that the random functions $Y_1,\ldots,Y_N$centered around 0 are part of a stationary times series of functional data which satisfies certain weak dependence conditions (compare chapter 16 of Horváth and Kokoszka (2010)). We again want to test for a change in the mean. The testing procedure is similar, but, as in the familiar scalar setting, we have to take into account that the variability of the sample mean $\bar{X}_N(t)$ depends on the kind of dependence of the data. In particular, the variability will be larger if the dependence is rather positive which is the more common situation in practice. This would lead to a larger number of false rejections of the above test procedure if we falsely assume independence. Therefore, we have to modify the test statistics accordingly. We follow the work of Hörmann and Kokoszka(2010), also described in Horváthand Kokoszka (2010).

As in the scalar case, the effect of dependence on mean tests can be summarized in the long-run variance. For a real-valued stationary time series $Z_t, -\infty < t < \infty$, with mean 0this quantity is the sum over all autocovariances

$$\sigma = \sum_{h=-\infty}^{\infty} \operatorname{cov}(Z_t, Z_{t+h}) = \sum_{h=-\infty}^{\infty} EZ_t, Z_{t+h}.$$

By stationarity, it does not depend on t. Equivalently, σ is the value of the power spectral density of the time series at 0.

The functional data enter the test statistic of the previous subsection only in form of the scores $\hat{y}_i = \left(\hat{y}_{i,1}, \ldots, \hat{y}_{i,d}\right)^T, i = 1, \ldots, N$,which is a sequence of d-dimensional random vectors. So, we need the long-run variance which now is a $d \times d$-covariance matrix, of ad-variate stationary time series $z_t, -\infty < t < \infty$, with mean 0 which is defined as

$$\mathbf{\Sigma} = \sum_{h=-\infty}^{\infty} \mathrm{E}\mathbf{z}_t\mathbf{z}_{t+h}^{\mathrm{T}}$$

To get an estimate of Σ, we estimate the autocovariances $\Gamma_h = Ez_t z_{t+h}^T$ by their empirical versions based on a sample $z_1, \dots, z_N$:

$$\hat{\Gamma}_h = \frac{1}{N}\sum_{i=1}^{N-h} z_i z_{i+h}^T$$

Then, we apply the windowing technique well known from one-dimensional spectral analysis to get with some suitable window width b_N depending on N

$$\widehat{\mathbf{\Sigma}}_N = \sum_{h=-N+1}^{N-1} K\big(\frac{h}{b_N}\big)\widehat{\Gamma}_h$$

K is a common kernel function which is bounded, symmetric around 0 and, for convenience, has a bounded support, say $[-1,+1]$. An example is the Bartlett kernel $K(u) = 1 - |u|$ for $|u| \leq 1$, and $K(u) = 0$, else. For $N, b_N \to \infty$ such that $b_N/N \to 0$ is a consistent estimate of Σ under some regularity conditions.

For getting an appropriate test statistic, set for $1 \leq k \leq N$

$$L_N(k) = \frac{1}{N}\left(\sum_{i=1}^{k}\hat{\mathbf{y}}_i - \frac{k}{N}\sum_{i=1}^{N}\hat{\mathbf{y}}_i\right)$$

Let $\hat{\Sigma}_N(\hat{y})$ denote the long-run variance estimate based on $z_i = \hat{y}_i$ and set

$$R_{N,d} = \frac{1}{N}\sum_{k=1}^{N} L_N^{\mathrm{T}}(k)\widehat{\mathbf{\Sigma}}_N^{-1}(\hat{\mathbf{y}})L_N^{\mathrm{T}}(k)$$

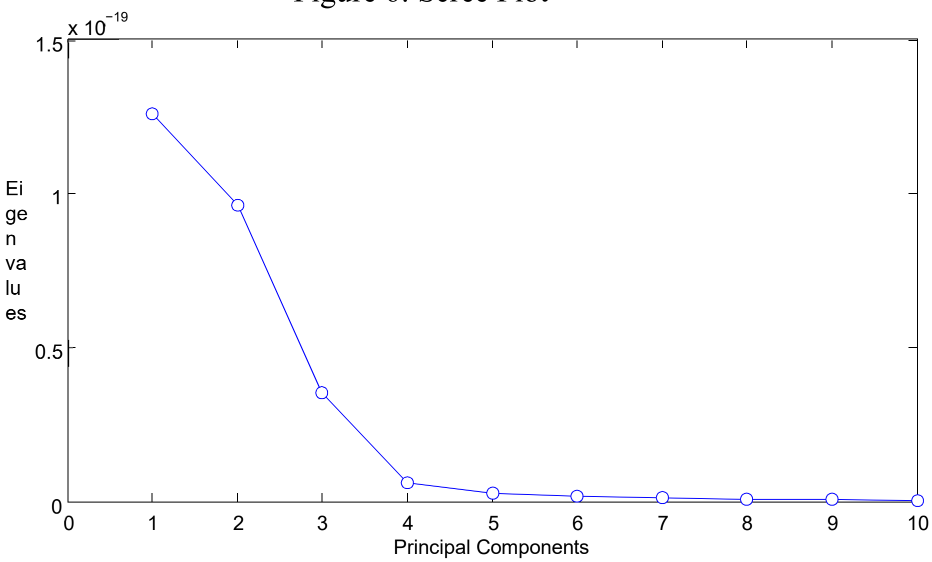
Note that for the diagonal matrix with entries $\hat{\lambda}_1, \dots, \hat{\lambda}_d$ replacing $\hat{\Sigma}_N(\hat{y})$, the integrand coincides with $\hat{T}_N(k)$ such that $R_{N,d}$ is a straightforward generalization of the test statistic $S_{N,d}$ of the previous section to the dependent case. The asymptotics do not change under the hypothesis and under the alternative as the effects of dependence are completely covered by the modification of the test statistic. Therefore, we may use the critical values from Table 6.1 of Horváthand Kokoszka (2010) for the changepoint test under dependence,too.

Note that in chapter 16 of Horváth and Kokoszka (2010) a slightly different version of the test statistic is considered, but it differs from ours only by replacing an integral by the corresponding Riemann sum which asymptotically is neglible.

4. Application to Stimulus Response Data

Before applying the changepoint test, we have to choose the number d of functional principal components entering the test statistic. This problem is closely related to the analogous problem in classical principal component analysis as a tool for dimension reduction, and we use a popular method, which is based on the scree plot, for selecting the number of relevant principal components based on the data.

Figure 6: Scree Plot



The screeplot shows how much each principal component contributes to the total variability of the data in decreasing order of importance. In the case of functional principal components, the contribution to total variability are just given by the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ of the covariance operator C introduced in subsection 3.1. Estimates $\hat{\lambda}_j$ are easily calculated using the fda package of R. Figure 6 shows the screeplot for the sample corresponding to the stimulus frequency 10 Hz. The screeplots of the other samples lookquite similar.

The idea of the scree plot is that we visually select the number d of principal components as the point where the curve dies off. Another more objective method for this purpose is requiring that the cumulative percentage of variance explained by the first d functional principal component has to be greater than some bound close to 100%, e.g. 85%. Based on Figure 6 and this rule, we decided to work with d = 4 functional principal components. They explain a cumulative percentage of variance of approximately 96%.

Assuming the data is independent and identically distributed, we applied the test described in subsection 3.1 to the data with stimulus frequency 1, 2, 5, 10 and 50 Hz. The data were adjusted to remove the artificial artifact, but not differenced. Table 1 reports the results obtained for significance level 0.05. Note that the asymptotic critical value, based on the relationship (3), does not depend on the sample size N due to an appropriate standardization of the test statistics $S_{N,d}$such that it is the same for all stimulus frequencies.

Table 1: Test for change in the mean function (i.i.d. Test)

α = 0.05, d=4, Asymptotic crit. value=1.239675

	1Hz	2Hz	5Hz	10Hz	50Hz
N	60	120	300	600	3000
Test statistic	2.0872	3.8249	8.5994	54.7244	212.0775

In all cases, a changepoint was detected as the values of the test statistic all exceedthe critical value.

Once the changepoint was detected, we estimated it using (4). Then, we split the sample and applied the test repeatedly until no further changepoints were detected. In Table 2 we list the detected changepoints in order of significance for the frequencies 1, 2, 5 and 10 Hz. These will be used for comparison with the changepoints for the dependent case. The changepoints are listed here as number of observed stimulus-response curve inthe sample and not as physical time.

Table 2: Changepoints in order of significance (i.i.d. Test)

Change points in order of significance (i.i.d. Test)

Frequency	Change Points						
1Hz	20						
2Hz	70	100					
5Hz	155	85					
10Hz	361	164	62	10	472	396	547

We also carried out the test for a change in the mean using the differenced data. As expected, for all frequencies no changepoint was detected which implies that these dataapproximately have a constant mean.

As discussed in subsection 3.2, the test of subsection 3.1, which is based on the assumption of independence, is known to give wrong results when the data show some dependency, likely too many rejections of the hypothesis. As we suspected dependence in our data which are response curves measured subsequently on the same cell, we tested for dependence. We carried out a Portmanteau test presented by Gabrys and Kokoszka (2007) for testing the hypothesis H_0 of independence of the curve data $X_1,\ldots,X_N$against an open ended alternative of lack of independence or of sameness of distributions. The corresponding test statistic is asymptotically chi-square distributed under the null hypothesis, such that critical values are well-known. The main assumption of the test is the existence of fourth moments of the observations which is likely be satisfied looking at the data. Also, the data should be stationary which of course is not true if the means are changing. Therefore, we applied the test to the differenced data $Y_j$given by (1). The results of the test are given in Table 3. In all cases the assumption of independence is rejected such that our data aregenuine functional time series.

Table 3: Portmanteau Test

α = 0.05, d=4, Asymptotic crit. value=67.5050

	1Hz	2Hz	5Hz	10Hz	50Hz
N	59	119	299	599	
Test statistic	176.3522	313.8736	334.5219	552.3081	2574.5181

As the data are likely dependent, the previous application of the test of subsection 3.1 is not justified. Therefore, we dropped the assumption of independence and applied the more complex test of Hӧrmann and Kokoszka (2010) described in subsection 3.2. The results of the tests are reported in Table 4; in all case we again detect a change in the mean on thesignificance level 0.05. However, the values of the test statistics are generally smaller. As the asymptotic distribution of the statistics $S_{N,d}$and $R_{N,d}$are identical, this means thatthe hypothesis is not so strongly rejected as if we falsely use the test for independent data.

Note that, as under the incorrect assumption of i.i.d. curve data, the test taking into account dependence also accepts the hypothesis of no change for all stimulus frequencies if we apply it to the differenced dataY_i.

Table 4: Test for change in the mean function (Dependent Test)

α = 0.05, d=4, Asymptotic crit. value=1.239675

	1Hz	2Hz	5Hz	10Hz	50Hz	
N	60	120	300	600	3000	6000
Test statistic	1.5847	2.0715	3.6859	8.6769	32.6208	

The differences between the two tests of subsections 3.1 and 3.2 are more striking once we apply it repeatedly to the split subsamples in search of more than one changepoint.Table 5 gives the change points in order of their significance based on the changepoint test for dependent data.

Table 5: Changepoints in order of significance (Dependent Test)

Frequency	Change Points								
1Hz	20								
2Hz	74								
5Hz	155								
10Hz	359	163	62	472	389				
50Hz	2067	1213	679	288	182	542	358	987	1081
	1787	1632	1924	2459 2330		2748	2591	2830	

Comparing the results to those from Table 2, we see that the test of subsection 3.1 for i.i.d. data detects many false changepoints as a result of failure to account for the long-run variance. Also, it is noticeable as expected, that with increasing frequency of the stimulus there are more changepoints. This can be attributed to the fact that at high frequency the cell does not have enough time to recover and go back to itsresting state before the next stimulus is given.

5. Conclusion

In this paper, we applied tests from functional data analysis to illustrate their merit in making use of the full information in stimulus response curve data. In particular, we showed that the subsequent detrended curve data are dependent. Using an appropriate changepoint test which takes into account the dependence, we were able to show that the originalcurve data showed several changes in the mean response curve throughout the experiment.

Our findings are in accordance with other statistical analyses of the same data. E.g., looking only at the univariate response latencies, i.e. the time span between stimulus and start of the response, we found an increasing trend which also was not homogeneous but showed changepoints between periods of rapid increase and periods of almost constancy.

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