

EXPANSION FORMULAS FOR EUROPEAN QUANTO OPTIONS IN A LOCAL VOLATILITY FX-LIBOR MODEL

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ABSTRACT. We develop an expansion approach for the pricing of European quanto options written on LIBOR rates (of a foreign currency). We derive the dynamics of the system of foreign LIBOR rates under the domestic forward measure and then consider the price of the quanto option. In order to take the skew/smile effect observed in fixed income and FX markets into account, we consider local volatility models for both the LIBOR and the FX rate. Because of the structure of the local volatility function, a closed form solution for quanto option prices does not exist. Using expansions around a proxy related to log-normal dynamics, we derive approximation formulas of Black–Scholes type for the price, that have the benefit of giving very rapid numerical procedures. Our expansion formulas have the major advantage that they allow for an accurate estimation of the error, using Malliavin calculus, which is directly related to the maturity of the option, the payoff, and the level and curvature of the local volatility function. These expansions also illustrate the impact of the quanto drift adjustment, while the numerical experiments show an excellent accuracy.

1. INTRODUCTION

We are interested in the pricing of European quanto options on LIBOR rates. These correspond to a type of derivative in which the underlying rate is denominated in one currency (foreign currency) but the payment is made in another currency (domestic currency). Such products are attractive for speculators and investors who wish to have exposure to a foreign asset, but without the corresponding exchange rate risk. Think, for instance, of a Euro-based investor who is seeking exposure on the GBP LIBOR rate, but does not want to be exposed to changes of the GBP/EUR foreign exchange rate. A European quanto option on the GBP LIBOR rate is a very suitable financial product for her, as it has the payoff of a standard non-quanto option on the GBP LIBOR rate and converts the payout with a guaranteed rate of 1 from GBP into Euro at maturity.

In an arbitrage-free framework, the pricing of quanto options can be performed under the domestic forward measure. In order to express the dynamics of the underlying LIBOR rate under this pricing measure, one has to apply Girsanov’s theorem, leading to a drift term which depends on the volatility of the LIBOR rate, on the volatility of the FX rate, and on the correlation between the LIBOR and the FX rate. This drift term leads to an adjustment in the pricing that is referred to as *quanto adjustment* and falls into the more general category of what is called in mathematical finance *convexity adjustment*.

This class of contracts, termed *exotic European options* by [Pellser \(2000\)](#), are widely traded over the counter (OTC). The correct pricing and risk management of European quanto options constitutes an important issue in the financial industry. The consideration of the market skew/smile for interest rates and FX rates is fundamental for a correct valuation of European quanto derivatives as discussed in [Romo \(2012\)](#). In reference textbooks and articles, see *e.g.* [Musielka and Rutkowski \(2005\)](#), [Brigo and Mercurio \(2006\)](#) or [Reiner \(1992\)](#), a simplified Black–Scholes model is considered in order to obtain analytical formulas. A similar practical approach is commonly used in the financial industry; see Section 5.2 or *e.g.* [Romo \(2012\)](#) and [Christoffersen and Jacobs \(2004\)](#) for more details. However, it does not take into account properly the skew/smile effects of the underlying assets in the quanto drift adjustment. These issues with the commonly used approach and the importance of incorporating the skew/smile properly in the valuation of

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European quanto options are studied and discussed in *e.g.* [Romo \(2012\)](#), [Jäckel \(2016\)](#), [Giese \(2012\)](#) or [Vong and Rojas-Carulla \(2014\)](#).

Local volatility models, either parametric or non-parametric, see *e.g.* [Derman and Kani \(1998\)](#); [Dupire \(1994\)](#); [Jäckel \(2008\)](#); [Rubinstein \(1994\)](#) or [Cox \(1975\)](#), usually capture the surface of implied volatilities more precisely than other approaches, such as stochastic volatility models; see *e.g.* [Ren, Madan, and Qian \(2007\)](#) or [Romo \(2012\)](#) for discussions. Moreover, the findings in [Romo \(2012\)](#) or [Hull and Suo \(2002\)](#) indicate that the local volatility model can be a correct approach to price European quanto derivatives in the presence of volatility skew/smile.

Motivated by the discussions above, we propose to evaluate European quanto options in a general local volatility framework. Because of its generality, it is often difficult to get analytical formulas for pricing, especially in a high-dimensional case. In general, the effective pricing requires the use of a numerical method, based either on PDE (partial differential equation) techniques or Monte Carlo simulations, which can be prohibitively time-consuming for real-time applications. Only in very few cases does one have closed-form formulas; *cf.* [Albanese, Campolieti, Carr, and Lipton \(2001\)](#). In the case of homogeneous volatility, singular perturbation techniques ([Hagan and Woodward \(1999\)](#)) have been used to obtain asymptotic expressions for the price of vanilla options (call, put). Implied volatility formulas are derived using asymptotic expansion methods for short maturities, as in [Berestycki, Busca, and Florent \(2002\)](#), [Henry-Labordère \(2005\)](#) and [Albanese et al. \(2001\)](#). In a more general diffusion setting, approximations of the density function and option prices are derived based on the small disturbance asymptotics, see *e.g.* [Kunitomo and Takahashi \(2004\)](#); [Yoshida \(1992\)](#) or [Takahashi \(1995, 1999\)](#) or [Takahashi \(2015\)](#) for a review. By adapting the singular perturbation method used in [Hagan and Woodward \(1999\)](#), several authors have developed expansion formulas for the density function of the underlying process and option prices in a general local volatility model; see *e.g.* [Pagliarani and Pascucci \(2012\)](#) and [Foschi, Pagliarani, and Pascucci \(2013\)](#).

The purpose of the present article is to provide simple and accurate approximation formulas for quanto options in a general local volatility model. Towards this end, we apply the perturbation method using a proxy introduced in [Benhamou, Gobet, and Miri \(2009\)](#). This method has been applied and extended in many directions, see *e.g.* [Benhamou, Gobet, and Miri \(2010b, 2012\)](#), [Gobet and Miri \(2014\)](#), [Gobet and Hok \(2014\)](#) and [Gobet and Bompis \(2014\)](#). We derive expansion formulas, which are of Black–Scholes type, and develop the analysis using Malliavin calculus, to provide accurate estimations of the errors. We believe that rigorous error estimates are of prime importance because the accuracy of our expansion formulas depends on the regularity of the payoff function. Once done, this brings confidence in the derived expansions and sheds light on the needed assumptions; see our main results in [Theorems 3.6](#) and [3.7](#).

A major advantage of our expansion formulas is that they clearly illustrate the impact of the quanto drift adjustment and provide very rapid numerical procedures for its implementation. The numerical tests, see [Section 5](#), show that our formulas constitute a very accurate approximation. Our interest in such problems was motivated by specific applications to European quanto derivatives on LIBOR rates, hence we specialize our study to that setting. However, the approximation methodology and results could be applied to other financial assets as well. Moreover, we focus on single-curve LIBOR models as they constitute the basis for multi-curve models. The extension to multiple curves is straightforward given the analysis and results of the present paper, however it is also very tedious.

This article is structured as follows: [Section 2](#) introduces local volatility models for simultaneously modeling FX and LIBOR rates, as well as quanto options. [Section 3](#) outlines the approach to quanto pricing via expansions around a proxy model and states the main results, which are second and third order expansions for the prices of quanto options. [Section 4](#) provides an error analysis and the derivation of the second order expansion formula, while [Section 5](#) provides numerical results. Finally, the Appendices contains some auxiliary results and the derivation of the third order expansion formula.

2. FX-LIBOR MODELS AND EUROPEAN QUANTO OPTIONS

2.1. A local volatility FX-LIBOR framework. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}_N)$ denote a filtered probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T_N]}$ satisfies the usual conditions and T_N denotes a finite time horizon. Let also $\mathcal{T} = \{0 = T_0 < T_1 < \dots < T_N\}$ denote a discrete tenor structure where $\delta_i = T_{i+1} - T_i$ is the accrual fraction for the period $[T_i, T_{i+1}]$, and define $\mathcal{I} = \{1, \dots, N\}$. The dates $(T_i)_{i \in \mathcal{I}}$ correspond to maturity dates of traded instruments.

We assume the existence of an arbitrage-free system of domestic and foreign zero coupon bonds, denoted respectively $(B(\cdot, T_i))_{i \in \mathcal{I}}$ and $(B^f(\cdot, T_i))_{i \in \mathcal{I}}$. We further consider domestic and foreign forward martingale measures, denoted by $(\mathbb{Q}_i)_{i \in \mathcal{I}}$ and $(\mathbb{Q}_i^f)_{i \in \mathcal{I}}$, where the corresponding zero coupon bond acts as the numeraire for each forward measure. Let $W = (W_t)_{t \in [0, T_N]}$ and $W^f = (W_t^f)_{t \in [0, T_N]}$ denote standard d -dimensional Brownian motions relative to the domestic and foreign terminal forward measures \mathbb{Q}_N and \mathbb{Q}_N^f respectively.

Let $(L_i)_{i \in \mathcal{I}}$ and $(L_i^f)_{i \in \mathcal{I}}$ denote the domestic and foreign forward LIBOR rates, *i.e.* the discretely compounded forward rates for investing in the time period $[T_i, T_{i+1}]$ in the domestic and foreign market. Their relation to zero coupon bonds is classically given by

$$1 + \delta_i L_i(t) = \frac{B(t, T_i)}{B(t, T_{i+1})} \quad \text{and} \quad 1 + \delta_i L_i^f(t) = \frac{B^f(t, T_i)}{B^f(t, T_{i+1})}. \quad (2.1)$$

The dynamics of the system of foreign LIBOR rates $(L_i^f)_{i \in \mathcal{I}}$ is provided by a local volatility model of the form

$$dL_i^f(t) = L_i^f(t) \lambda_i(t, L_i^f(t)) dW_t^{f, i+1}, \quad (2.2)$$

where the \mathbb{Q}_{i+1}^f -Brownian motion is related to the terminal Brownian motion via

$$W^{f, i+1} = W^{f, N} - \sum_{k=i+1}^N \int_0^{\cdot} \frac{\delta_k L_k^f(t)}{1 + \delta_k L_k^f(t)} \lambda_k(t, L_k^f(t)) dt, \quad (2.3)$$

for all $t \in [0, T_{i+1}]$. The functions $\lambda_i : [0, T_i] \times \mathbb{R} \rightarrow \mathbb{R}_+^d$, $i \in \mathcal{I}$, are continuous, deterministic and satisfy a suitable linear growth condition, *cf.* [Brigo and Mercurio \(2006, §10.3\)](#). They represent the local volatility of the foreign forward LIBOR rate L_i^f .

Let $(X(t))_{t \in [0, T_N]}$ denote the foreign exchange (FX) rate expressed in terms of units of domestic currency per unit of foreign currency. The FX forward rate for settlement at time T_i , denoted by $(X_i(t))_{t \in [0, T_i]}$, is defined by no-arbitrage arguments and provided by

$$X_i(t) = \frac{B^f(t, T_i) X(t)}{B(t, T_i)}, \quad (2.4)$$

for all $t \in [0, T_i]$. The FX forward rate is, per definition, a \mathbb{Q}_i -martingale, and we assume it follows again a local volatility model of the form

$$dX_i(t) = X_i(t) \sigma_i(t, X_i(t)) dW_t^i, \quad (2.5)$$

where W^i is the \mathbb{Q}_i -Brownian motion and $\sigma_i : [0, T_i] \times \mathbb{R} \rightarrow \mathbb{R}_+^d$, $i \in \mathcal{I}$, is a continuous, deterministic function satisfying a suitable linear growth condition, and represents the local volatility of the FX forward rate.

The domestic and foreign forward Brownian motions are related via

$$W^{f, i} = W^i - \int_0^{\cdot} \sigma_i(t, X_i(t)) dt, \quad (2.6)$$

for all $i \in \mathcal{I}$, and this equation together with (2.3) determines also the relations between the domestic forward Brownian motions; see [Schlögl \(2002\)](#) for the details (in particular Fig. 2).

Considering the logarithms of the LIBOR and the FX rate, denoted by $Y = \ln L$ and $Z = \ln X$, the system of SDEs (2.10) takes the form

$$\begin{cases} dY_t = \alpha(t, Y_t, Z_t)dt + \lambda(t, Y_t)dW_t^L, & Y_0 = \ln L_0 = y_0 \\ dZ_t = \beta(t, Z_t)dt + \sigma(t, Z_t)dW_t^X, & Z_0 = \ln X_0 = z_0, \\ \langle W^L, W^X \rangle = \rho, \end{cases} \quad (3.2)$$

where

$$\begin{cases} \alpha(t, y, z) = -[\frac{1}{2}\lambda^2(t, y) + \rho\lambda(t, y)\sigma(t, z)] \\ \lambda(t, y) = \lambda_i(t, e^y) \\ \beta(t, z) = -\frac{1}{2}\sigma^2(t, z) \\ \sigma(t, z) = \sigma_{i+1}(t, e^z). \end{cases} \quad (3.3)$$

Moreover, the price of a quanto option with generic payoff function $h : \mathbb{R} \rightarrow \mathbb{R}_+$ is provided by

$$\mathbb{Q}\mathbb{O}_h(T) = \delta B(0, T) \mathbb{E}[h(Y_T)]. \quad (3.4)$$

Taking $h(y) = (e^y - K)^+$, we recover the quanto caplet in (2.8)–(2.9).

3.1. Closed-form formula under log-normal dynamics. Assume that the local volatility coefficients are deterministic, time-dependent functions, in particular $\lambda(t, y) \equiv \lambda(t, y_0)$ and $\sigma(t, z) \equiv \sigma(t, z_0)$, for all $t \in [0, T]$. Then, Y_T follows a normal distribution and we can derive a closed-form formula for the price of the quanto caplet defined in (2.8).

Indeed, using standard results from stochastic calculus, we have that

$$Y_T = y_0 - \frac{1}{2}\Lambda(T) - \Sigma(T) + \int_0^T \lambda(t, y_0)dW_t^L, \quad (3.5)$$

where

$$\Lambda(T) = \int_0^T \lambda^2(t, y_0)dt \quad \text{and} \quad \Sigma(T) = \rho \int_0^T \lambda(t, y_0)\sigma(t, z_0)dt. \quad (3.6)$$

Thus, exactly as in [Brigo and Mercurio \(2006\)](#), the price of the quanto caplet is given by the following, Black–Scholes type, formula:

$$\begin{aligned} \mathbb{Q}\mathbb{C}^{BS}(T, K) &= \delta B(0, T) \mathbb{E}[(e^{Y_T} - K)^+] \\ &= \delta B(0, T) \mathbb{C}^{BS}(y_0) \end{aligned} \quad (3.7)$$

with

$$\mathbb{C}^{BS}(y_0) = e^{y_0 - \Sigma(T)} \Phi(d_1) - e^k \Phi(d_2), \quad (3.8)$$

where Φ denotes the cdf of the standard normal distribution, $k = \ln K$, and

$$d_1 = \frac{y_0 - k - \Sigma(T) + \frac{1}{2}\Lambda(T)}{\sqrt{\Lambda(T)}} \quad \text{and} \quad d_2 = d_1 - \sqrt{\Lambda(T)}. \quad (3.9)$$

3.2. Expansion formulas under general dynamics. The aim of this subsection is to provide expansion formulas that approximate the price of a quanto option when we consider general local volatility dynamics for the LIBOR and the FX rate. Let us first introduce some notation, and some assumptions that allow us to derive these formulas.

Assumption (\mathbb{R}_n). The volatility functions $\lambda(\cdot, y)$ and $\sigma(\cdot, z)$ are of class C^n in y and z respectively, for some $n \in \mathbb{N}$. In addition, these functions and their derivatives are uniformly bounded.

Let us introduce the following constants

$$M_1^\lambda := \max_{1 \leq i \leq n} \sup_{(t, y) \in [0, T] \times \mathbb{R}} |\partial_y^i \lambda(t, y)|, \quad M_1^\sigma := \max_{1 \leq i \leq n} \sup_{(t, z) \in [0, T] \times \mathbb{R}} |\partial_z^i \sigma(t, z)| \quad (3.10)$$

$$M_0^\lambda := \max\{M_1^\lambda, \sup_{(t, y) \in [0, T] \times \mathbb{R}} |\lambda(t, y)|\}, \quad M_0^\sigma := \max\{M_1^\sigma, \sup_{(t, z) \in [0, T] \times \mathbb{R}} |\sigma(t, z)|\} \quad (3.11)$$

and also

$$M_1 := \max\{M_1^\lambda, M_1^\sigma\}, \quad M_0 := \max\{M_0^\lambda, M_0^\sigma\}. \quad (3.12)$$

Let us denote by \mathcal{H} the space of functions with growth being at most exponential. In other words, a function h belongs to \mathcal{H} if $|h(x)| \leq c_1 e^{c_2|x|}$, for any x , for two positive constants c_1 and c_2 . Moreover, let $h^{(k)}$ denote the k -th derivative of the function h .

We will separate our analysis according to the smoothness of the payoff function, and distinguish between two cases:

Assumption (\mathbb{S}_1). The payoff function h belongs to $C_0^\infty(\mathbb{R}, \mathbb{R})$, the space of real-valued infinitely differentiable functions with compact support.

Assumption (\mathbb{S}_2). The payoff function h is almost everywhere differentiable. In addition h and $h^{(1)}$ belong to \mathcal{H} .

Remark 3.1. The first assumption corresponds to (idealized) smooth payoff functions, while the second one corresponds to call and put options.

In real markets, the correlation between the forward LIBOR rate and the forward FX rate is typically not very large. As an example, the empirical study in [Boenkost and Schmidt \(2003\)](#) found its value in the range $[-0.2, 0.2]$. Therefore, the following assumption is consistent with real market data.

Assumption (\mathbb{RHO}). The correlation between the forward LIBOR rate and the forward FX rate is not perfect, *i.e.*

$$|\rho| < 1. \quad (3.13)$$

In order to perform the infinitesimal analysis in the error estimates, we rely on smoothness properties which are not provided by the payoff functions, but rather by the law of the underlying stochastic models; this is related to Malliavin calculus. The following ellipticity assumption on the volatility of the forward LIBOR combined with Assumption (\mathbb{R}_4)—*i.e.* Assumption (\mathbb{R}_n) with $n = 4$ —guarantees that sufficient smoothness is available.

Assumption (\mathbb{ELL}). The volatility of the forward LIBOR rate λ does not vanish and for a positive constant C_E , one has

$$1 \leq \frac{\|\lambda\|_\infty}{\lambda_{\inf}} \leq C_E, \quad (3.14)$$

where $\|g\|_\infty = \sup_{(t,y) \in [0,T] \times \mathbb{R}} |g(t,y)|$ and $\lambda_{\inf} = \inf_{(t,y) \in [0,T] \times \mathbb{R}} |\lambda(t,y)|$.

We consider now the following ‘proxy’ or ‘Black–Scholes’ processes:

$$\begin{cases} dY_t^0 &= \alpha(t, y_0, z_0)dt + \lambda(t, y_0)dW_t^L, & Y_0^0 &= y_0, \\ dZ_t^0 &= \beta(t, z_0)dt + \sigma(t, z_0)dW_t^X, & Z_0^0 &= z_0, \\ \langle W^L, W^X \rangle &= \rho, \end{cases} \quad (3.15)$$

and introduce a family of parametrized processes (Y^η, Z^η) , for $\eta \in [0, 1]$, via the system of SDEs:

$$\begin{cases} dY_t^\eta &= \alpha(t, \eta Y_t^\eta + (1-\eta)y_0, \eta Z_t^\eta + (1-\eta)z_0)dt \\ &\quad + \lambda(t, \eta Y_t^\eta + (1-\eta)y_0)dW_t^L, & Y_0^\eta &= y_0, \\ dZ_t^\eta &= \beta(t, \eta Z_t^\eta + (1-\eta)z_0)dt \\ &\quad + \sigma(t, \eta Z_t^\eta + (1-\eta)z_0)dW_t^X, & Z_0^\eta &= z_0, \\ \langle W^L, W^X \rangle &= \rho. \end{cases} \quad (3.16)$$

Setting $\eta = 1$, we recover the dynamics of the local volatility model in (3.2) since $Y_t^1 = Y_t$ and $Z_t^1 = Z_t$, while for $\eta = 0$ we recover the Black–Scholes proxy in (3.15).

Assumption (\mathbb{R}_4) yields that, almost surely for any $t \in [0, T]$, $\frac{\partial}{\partial \eta}(Y_t^\eta, Z_t^\eta)$ is C^3 with respect to η ; see *e.g.* [Bell \(2006\)](#) or [Kunita \(1997\)](#). Setting $Y_{i,t}^\eta = \frac{\partial^i Y_t^\eta}{\partial \eta^i}$, $Z_{i,t}^\eta = \frac{\partial^i Z_t^\eta}{\partial \eta^i}$ and by a direct

differentiation of the SDEs (3.16), we get that

$$\begin{cases} dZ_{1,t}^\eta &= (\eta Z_{1,t}^\eta + Z_t^\eta - z_0)[\beta_z dt + \sigma_z dW_t^X], \\ dY_{1,t}^\eta &= [(\eta Y_{1,t}^\eta + Y_t^\eta - y_0)\alpha_y + (\eta Z_{1,t}^\eta + Z_t^\eta - z_0)\alpha_z]dt \\ &\quad + (\eta Y_{1,t}^\eta + Y_t^\eta - y_0)\lambda_y dW_t^L, \end{cases} \quad (3.17)$$

with $Y_{1,0}^\eta = Z_{1,0}^\eta = 0$, and

$$\begin{cases} dZ_{2,t}^\eta &= (2Z_{1,t}^\eta + \eta Z_{2,t}^\eta)[\beta_z dt + \sigma_z dW_t^X] \\ &\quad + (\eta Z_{1,t}^\eta + Z_t^\eta - z_0)^2[\beta_{zz} dt + \sigma_{zz} dW_t^X], \\ dY_{2,t}^\eta &= (2Y_{1,t}^\eta + \eta Y_{2,t}^\eta)[\alpha_y dt + \lambda_y dW_t^L] + (2Z_{1,t}^\eta + \eta Z_{2,t}^\eta)\alpha_z dt \\ &\quad + 2(\eta Y_{1,t}^\eta + Y_t^\eta - y_0)(\eta Z_{1,t}^\eta + Z_t^\eta - z_0)\alpha_{yz} dt \\ &\quad + [(\eta Y_{1,t}^\eta + Y_t^\eta - y_0)^2\alpha_{yy} + (\eta Z_{1,t}^\eta + Z_t^\eta - z_0)^2\alpha_{zz}]dt \\ &\quad + (\eta Y_{1,t}^\eta + Y_t^\eta - y_0)^2\lambda_{yy} dW_t^L, \end{cases} \quad (3.18)$$

with $Y_{2,0}^\eta = Z_{2,0}^\eta = 0$, and

$$\begin{cases} dZ_{3,t}^\eta &= (3Z_{2,t}^\eta + \eta Z_{3,t}^\eta)[\beta_z dt + \sigma_z dW_t^X] \\ &\quad + 3(2Z_{1,t}^\eta + \eta Z_{2,t}^\eta)(\eta Z_{1,t}^\eta + Z_t^\eta - Z_0)[\beta_{zz} dt + \sigma_{zz} dW_t^X] \\ &\quad + (\eta Z_{1,t}^\eta + Z_t^\eta - Z_0)^3[\beta_{zzz} dt + \sigma_{zzz} dW_t^X], \\ dY_{3,t}^\eta &= (3Y_{2,t}^\eta + \eta Y_{3,t}^\eta)[\alpha_y dt + \lambda_y dW_t^L] + (3Z_{2,t}^\eta + \eta Z_{3,t}^\eta)\alpha_z dt \\ &\quad + 3(\eta Y_{1,t}^\eta + Y_t^\eta - Y_0)(2Y_{1,t}^\eta + \eta Y_{2,t}^\eta)[\alpha_{yy} dt + \lambda_{yy} dW_t^L] \\ &\quad + 3(\eta Z_{1,t}^\eta + Z_t^\eta - Z_0)(2Z_{1,t}^\eta + \eta Z_{2,t}^\eta)\alpha_{zz} dt \\ &\quad + 3[(\eta Y_{1,t}^\eta + Y_t^\eta - Y_0)(2Z_{1,t}^\eta + \eta Z_{2,t}^\eta) \\ &\quad \quad + (\eta Z_{1,t}^\eta + Z_t^\eta - Z_0)(2Y_{1,t}^\eta + \eta Y_{2,t}^\eta)]\alpha_{yz} dt \\ &\quad + 3[(\eta Y_{1,t}^\eta + Y_t^\eta - Y_0)^2(\eta Z_{1,t}^\eta + Z_t^\eta - Z_0)\alpha_{yyz} \\ &\quad \quad + (\eta Y_{1,t}^\eta + Y_t^\eta - Y_0)(\eta Z_{1,t}^\eta + Z_t^\eta - Z_0)^2\alpha_{zzy}]dt \\ &\quad + (\eta Y_{1,t}^\eta + Y_t^\eta - Y_0)^3\alpha_{yyy} dt + (\eta Z_{1,t}^\eta + Z_t^\eta - Z_0)^3\alpha_{zzz} dt \\ &\quad + (\eta Y_{1,t}^\eta + Y_t^\eta - Y_0)^3\lambda_{yyy} dW_t^L, \end{cases} \quad (3.19)$$

with $Y_{3,0}^\eta = Z_{3,0}^\eta = 0$. Here, we have used the following shorthand notation for the first order derivatives of the coefficients of the SDEs

$$\begin{cases} \alpha_x &= \frac{\partial \alpha}{\partial x}(t, y, z) \Big|_{y=\eta Y_t^\eta + (1-\eta)y_0, z=\eta Z_t^\eta + (1-\eta)z_0}, \quad x \in \{y, z\}, \\ \lambda_y &= \frac{\partial \lambda}{\partial y}(t, y) \Big|_{y=\eta Y_t^\eta + (1-\eta)y_0}, \\ \beta_z &= \frac{\partial \beta}{\partial z}(t, z) \Big|_{z=\eta Z_t^\eta + (1-\eta)z_0}, \\ \sigma_z &= \frac{\partial \sigma}{\partial z}(t, z) \Big|_{z=\eta Z_t^\eta + (1-\eta)z_0}, \end{cases} \quad (3.20)$$

and analogously for higher order derivatives.

Let us now introduce the main tools of this method, which are expansions of the random variable and the payoff function of the quanto option around known values. In order to keep the notation simple, we set $Y_{i,t} = \frac{\partial^i Y_t^\eta}{\partial \eta^i} \Big|_{\eta=0}$, $Z_{i,t} = \frac{\partial^i Z_t^\eta}{\partial \eta^i} \Big|_{\eta=0}$. Then, by performing a Taylor expansion of Y_T around zero, we get that

$$Y_T = Y_T^0 + Y_{1,T} + \frac{1}{2}Y_{2,T} + \frac{1}{2} \int_0^1 Y_{3,T}^\eta (1-\eta)^2 d\eta. \quad (3.21)$$

The dynamics of the proxy model in (3.15) yield that Y_T^0 is a Gaussian random variable with mean m_T^0 and variance $\sqrt{V_T^0}$, where

$$m_T^0 = y_0 + \int_0^T \alpha(t, y_0, z_0) dt \quad \text{and} \quad V_T^0 = \int_0^T \lambda^2(t, y_0) dt. \quad (3.22)$$

Performing now a Taylor expansion of the payoff in (3.4) around $h(Y_T^0)$, we arrive at a formula of the following form:

$$\mathbb{E}[h(Y_T)] = \mathbb{E}[h(Y_T^0)] + \text{Corrections terms} + \text{Error}. \quad (3.23)$$

The first term $\mathbb{E}[h(Y_T^0)]$ constitutes the leading order contribution, it is explicitly known (via an analytical formula analogous to (3.7) for the payoff function h), but as an approximation alone is not accurate enough. Therefore, in the sequel we will derive correction terms in order to achieve better accuracy. These correction terms are represented as a combination of Greeks of the option price formula (3.7). Hence, the numerical evaluation of all these terms is straightforward, with a computational cost equivalent to the analytical formula (3.7).

3.3. Definitions and notation. Before providing the main results, let us introduce some definitions and notation that will be used in the sequel.

Definition 3.2 (Integral Operator). The integral operator ω is defined as follows: for any integrable function l , set

$$\omega(l)_t^T = \int_t^T l_u du \quad (3.24)$$

for $t \in [0, T]$. Similarly, for integrable functions (l_1, l_2) and $t \in [0, T]$ set

$$\omega(l_1, l_2)_t^T = \omega(l_1 \omega(l_2)_t^T)_t^T = \int_t^T l_{1,r} \left(\int_r^T l_{2,s} ds \right) dr. \quad (3.25)$$

This can be easily iterated to define $\omega(l_1, l_2, \dots, l_n)_t^T = \omega(l_1 \omega(l_2, \dots, l_n)_t^T)_t^T$.

Definition 3.3 (Greeks). Let h be an appropriate payoff function (such that the expression below makes sense). We set, for $i \geq 0$,

$$g_i^h(Y_T^0) = \frac{\partial^i}{\partial \epsilon^i} \mathbb{E}[h(Y_T^0 + \epsilon)] \Big|_{\epsilon=0}. \quad (3.26)$$

Remark 3.4 (Generic constants). We use the notation $A \leq_c B$ to assert that $A \leq cB$, where c is a positive constant depending on the model parameters, on M_0, M_1, T, C_E and on other universal constants. The constant c may change from line to line, but remains bounded when the model parameters go to 0.

Remark 3.5 (Notation for the coefficients). The coefficients $\alpha, \beta, \lambda, \sigma$ and their derivatives will be evaluated from now on at the initial values (y_0, z_0) , *i.e.* when we write $\alpha, \beta, \lambda, \sigma$ we mean $\alpha(\cdot, y_0, z_0), \beta(\cdot, z_0), \lambda(\cdot, y_0), \sigma(\cdot, z_0)$, and the same holds for their derivatives. We will sometimes also use the subscript t when we want to stress their dependence on time.

3.4. Main results. We are now ready to state the main results of this work, that provide second and third order expansions of an option price around the proxy model, thus making precise the formula in (3.23). The proofs are deferred to Section 4.

Theorem 3.6 (2nd order expansion in price). *Assume that conditions (\mathbb{R}_3) , (\mathbb{S}_2) , (ELL) and (RHO) are in force. Then, the second order expansion of the option price takes the form:*

$$\begin{aligned} \mathbb{E}[h(Y_T)] &= \mathbb{E}[h(Y_T^0)] + \omega(\lambda^2, \lambda_y \lambda)_0^T \left[\frac{1}{2} g_1^h(Y_T^0) - \frac{3}{2} g_2^h(Y_T^0) + g_3^h(Y_T^0) \right] \\ &\quad + \rho \left[g_1^h(Y_T^0) [\omega(\lambda \sigma, \lambda_y \lambda)_0^T + \frac{1}{2} (\omega(\lambda^2, \lambda_y \sigma)_0^T + \omega(\sigma^2, \lambda \sigma_z)_0^T)] \right. \\ &\quad \left. - g_2^h(Y_T^0) [\omega(\lambda \sigma, \lambda_y \lambda)_0^T + \omega(\lambda^2, \lambda_y \sigma)_0^T] \right] \\ &\quad + \rho^2 \left[g_1^h(Y_T^0) \omega(\lambda \sigma, \lambda_y \sigma)_0^T - g_2^h(Y_T^0) \omega(\lambda \sigma, \lambda \sigma_z)_0^T \right] + \text{Error}_2. \end{aligned} \quad (3.27)$$

Additionally, the error estimate is provided by

$$|\text{Error}_2| \leq c \left[\|h^{(1)}(Y_T^0)\|_2 + \int_0^1 \|h^{(1)}(\eta Y_T + (1-\eta)Y_T^0)\|_2 d\eta \right] \frac{M_0}{\lambda_{\inf}(1-\rho^2)} M_1 M_0^2 T^{\frac{3}{2}}. \quad (3.28)$$

Theorem 3.7 (3rd order expansion in price). *Assume that conditions (\mathbb{R}_4) , (\mathbb{S}_2) , (ELL) and (RHO) are in force. Then, the third order expansion of the option price takes the form:*

$$\mathbb{E}[h(Y_T)] = \mathbb{E}[h(Y_T^0)] + \sum_{j=1}^6 \gamma_{0,j,T} g_j^h(Y_T^0) + \sum_{i=1}^4 \gamma_{i,T} \rho^i + \text{Error}_3, \quad (3.29)$$

where

$$\begin{cases} \gamma_{0,1,T} &= \frac{1}{2}(A_{1,T} - A_{2,T} - A_{3,T}) - \frac{1}{2}B_{1,T} - \frac{1}{4}(B_{2,T} + B_{3,T}) \\ \gamma_{0,2,T} &= -\frac{3}{2}A_{1,T} + \frac{1}{2}(A_{2,T} + A_{3,T}) + \frac{7}{2}B_{1,T} + \frac{5}{4}(B_{3,T} + B_{2,T}) + \frac{1}{2}C_{33,T} + \frac{1}{4}C_{32,T} \\ \gamma_{0,3,T} &= A_{1,T} - 6B_{1,T} - 2(B_{3,T} + B_{2,T}) - \frac{3}{2}C_{32,T} - 3C_{33,T} \\ \gamma_{0,4,T} &= 3B_{1,T} + B_{2,T} + B_{3,T} + \frac{13}{4}C_{32,T} + \frac{13}{2}C_{33,T} \\ \gamma_{0,5,T} &= -3C_{32,T} - 6C_{33,T} \\ \gamma_{0,6,T} &= C_{32,T} + 2C_{33,T} \end{cases} \quad (3.30)$$

with

$$\begin{cases} A_{1,T} = \omega(\lambda^2, \lambda\lambda_y)_0^T & A_{2,T} = \omega(\lambda^2, \lambda\lambda_{yy})_0^T & A_{3,T} = \omega(\lambda^2, (\lambda_y)^2)_0^T \\ B_{1,T} = \omega(\lambda^2, \lambda\lambda_y, \lambda\lambda_y)_0^T & B_{2,T} = \omega(\lambda^2, \lambda^2, \lambda\lambda_{yy})_0^T & B_{3,T} = \omega(\lambda^2, \lambda^2, (\lambda_y)^2)_0^T \\ C_{32,T} = \omega(\lambda^2, \lambda\lambda_y, \lambda^2, \lambda\lambda_y)_0^T & C_{33,T} = \omega(\lambda^2, \lambda^2, \lambda\lambda_y, \lambda\lambda_y)_0^T. \end{cases} \quad (3.31)$$

The expressions for the coefficients $\gamma_{i,T}$, $i = 1, 2, 3, 4$, are provided respectively by (C.16), (C.22), (C.28) and (C.33) in Appendix C. Additionally, the error estimate is given by

$$|\text{Error}_3| \leq c \left[\|h^{(1)}(Y_T^0)\|_2 + \int_0^1 \|h^{(1)}(\eta Y_T + (1-\eta)Y_T^0)\|_2 d\eta \right] \frac{M_0^5 M_1}{(\lambda_{\inf}(1-\rho^2))^2} T^2. \quad (3.32)$$

Remark 3.8 (Sanity check). Let $\rho = 0$, then the quanto (drift) adjustment in the LIBOR SDE (3.2) vanishes, and we recover the second and third order approximation formulas given respectively by Theorems 2.1 and 2.3 in Benhamou, Gobet, and Miri (2010a).

If $\rho \neq 0$, the second and third order expansion formulas (3.27) and (3.29) provide some information about the impact of the quanto (drift) adjustment in the option prices, in terms of a polynomial function of the correlation ρ .

Consider a call option with payoff $h(y) = (e^y - e^k)_+$, then the theorems above provide an approximation formula for its price in log variables. In that case, $\mathbb{E}[h(Y_T^0)] = \mathbb{C}^{BS}(y_0)$ corresponds to the Black–Scholes price given by (3.7). In order to compute the correction terms, we need to calculate the derivatives of $\mathbb{C}^{BS}(y_0)$ w.r.t. y_0 . Below is a useful lemma allowing to calculate them in a systematic way using Hermite polynomials.

Lemma 3.9. *Let $n \geq 1$, then we have*

$$\frac{\partial^n \mathbb{C}^{BS}(y_0)}{\partial y_0^n} = e^{y_0 - \Sigma(T)} \left[\Phi(d_1) + 1_{\{n \geq 2\}} \Phi'(d_1) \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^{j-1} \frac{H_{j-1}(d_1)}{(\lambda\sqrt{T})^j} \right], \quad (3.33)$$

where H_j , $j \in \mathbb{N}$, denotes the Hermite polynomials defined as

$$H_j(x) = (-1)^j e^{\frac{x^2}{2}} \partial_x^n (e^{-\frac{x^2}{2}}), \quad j \in \mathbb{N}. \quad (3.34)$$

The proof is provided in Appendix A.

4. ANALYSIS AND PROOFS

This section provides the derivation of the expansion formulas for quanto pricing presented in Theorems 3.6 and 3.7, as well as an analysis of the corresponding error terms. After some preliminary results, the expansion formulas and the corresponding error estimates for the second and third order expansions are presented in Section 4.2. The derivation of the Greeks for the second order expansion is presented in Section 4.3, while the details for the Greeks of the third order expansion are deferred to the appendix for the sake of brevity.

4.1. Auxiliary results. We start with some results that are useful for the subsequent error analysis. The L^p -estimates follow from the work of Benhamou et al. (2010a, Theorem 5.1), thus their proof is omitted. As usual, the L^p -norm of a real random variable Z is provided by $\|Z\|_p = (\mathbb{E}[|Z|^p])^{\frac{1}{p}}$, $p \geq 1$.

Lemma 4.1 (L^p -estimates). *Assume that condition (\mathbb{R}_4) is in force. Then, for all $p \geq 1$ and $i = 1, 2, 3$, we have*

$$\sup_{t \in [0, T], \eta \in [0, 1]} \|Z_t^\eta - z_0\|_p \leq c M_0 \sqrt{T}, \quad \sup_{t \in [0, T], \eta \in [0, 1]} \|Y_t^\eta - y_0\|_p \leq c M_0 \sqrt{T}, \quad (4.1)$$

$$\sup_{t \in [0, T], \eta \in [0, 1]} \|Z_{i,t}^\eta\|_p \leq c M_1 M_0^i T^{\frac{i+1}{2}}, \quad \sup_{t \in [0, T], \eta \in [0, 1]} \|Y_{i,t}^\eta\|_p \leq c M_1 M_0^i T^{\frac{i+1}{2}}. \quad (4.2)$$

The following lemmata are used repeatedly in order to derive the analytical formulas in Theorems 3.6 and 3.7. An application of Itô's lemma to $(\int_t^T f_s ds) Z_t$ yields the following result.

Lemma 4.2. *Let f be a continuous (or piecewise continuous) function and Z be a continuous semimartingale with $Z_0 = 0$. Then*

$$\int_0^T f_t Z_t dt = \int_0^T \left(\int_t^T f_s ds \right) dZ_t = \int_0^T \omega(f)_t^T dZ_t. \quad (4.3)$$

The lemma below follows directly from the duality relationship in the Malliavin calculus (see e.g. Nualart (2005, Lemma 1.2.1, p.25)) and by identifying Itô's integral and the Skorohod operator for adapted integrands.

Lemma 4.3. *Let u be a square integrable, progressively measurable process and assume h satisfies (\mathbb{S}_1) . Then, for any $i \geq 0$, it holds:*

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T u_t dW_t^\alpha \right) h^{(i)} \left(\int_0^T \lambda(t, y_0) dW_t^L \right) \right] \\ &= \mathbb{E} \left[\left(\int_0^T u_t \lambda(t, y_0) d\langle W^\alpha, W^L \rangle_t \right) h^{(i+1)} \left(\int_0^T \lambda(t, y_0) dW_t^L \right) \right] \end{aligned} \quad (4.4)$$

with $\alpha \in \{L, X\}$ and $h^{(i)}(x) = \frac{d^i}{dx^i} h(x)$, $i \in \mathbb{N}$. Moreover, if u and $\langle W^\alpha, W^L \rangle$ are deterministic, then

$$\mathbb{E} \left[\left(\int_0^T u_t dW_t^\alpha \right) h^{(i)} \left(\int_0^T \lambda(t, y_0) dW_t^L \right) \right] = \int_0^T u_t \lambda(t, y_0) d\langle W^\alpha, W^L \rangle_t g_{i+1}^{\tilde{h}}(Y_T^0), \quad (4.5)$$

where $\tilde{h}(x) = h(x - m_T^0)$.

4.2. Price expansions and error estimates. We are now ready to provide the details in the derivation of the expansion formulas and the corresponding error estimates. We start with the analysis of the second order approximation, and divide the proof of Theorem 3.6 in several steps. First, we assume that the payoff h is smooth and establish error estimates that depend only on $h^{(1)}$, the first derivative of h . To this end, we use Malliavin calculus and provide tight estimates on the Malliavin derivatives of the parametrized process. Then, we can approximate h under (\mathbb{S}_2) by a sequence of smooth payoffs using a density argument. This last step is standard by now, hence we omit it for the sake of brevity.

4.2.1. Second order error analysis. As outlined in the previous section, we perform first a Taylor expansion of Y_T around Y_T^0 , that yields

$$Y_T = Y_T^0 + Y_{1,T} + \int_0^1 Y_{2,T}^\eta (1 - \eta) d\eta, \quad (4.6)$$

then another Taylor expansion for the smooth payoff h , and then take expectations. Thus we obtain

$$\begin{aligned} \mathbb{E}[h(Y_T)] &= \mathbb{E}[h(Y_T^0)] + \mathbb{E}[h^{(1)}(Y_T^0)(Y_T - Y_T^0)] \\ &\quad + \mathbb{E}\left[(Y_T - Y_T^0)^2 \int_0^1 h^{(2)}(\eta Y_T + (1 - \eta)Y_T^0)(1 - \eta) d\eta\right]. \end{aligned} \quad (4.7)$$

Using (4.6), (4.7) can be written as

$$\mathbb{E}[h(Y_T)] = \mathbb{E}[h(Y_T^0)] + \mathbb{E}[h^{(1)}(Y_T^0)Y_{1,T}] + \text{Error}_2. \quad (4.8)$$

where

$$\text{Error}_2 = \mathbb{E}[h^{(1)}(Y_T^0)R_T^{1,Y}] + \mathbb{E}\left[(R_T^{0,Y})^2 \int_0^1 h^{(2)}(\eta Y_T + (1 - \eta)Y_T^0)(1 - \eta) d\eta\right] \quad (4.9)$$

with

$$R_T^{0,Y} = \int_0^1 Y_{1,T}^\eta d\eta \quad \text{and} \quad R_T^{1,Y} = \int_0^1 Y_{2,T}^\eta (1 - \eta) d\eta. \quad (4.10)$$

Using (4.2) with $i = 2$ in Lemma 4.1 and the Cauchy–Schwarz inequality, the first term in (4.9) is estimated as

$$|\mathbb{E}[h^{(1)}(Y_T^0)R_T^{1,Y}]| \leq c \|h^{(1)}(Y_T^0)\|_2 M_1 M_0^2 T^{\frac{3}{2}}. \quad (4.11)$$

The second term in (4.9) requires some additional work because of $h^{(2)}$. We use the integration-by-parts formula in the Malliavin calculus to write it using $h^{(1)}$ only. For this, we rely on Lemma B.2 and refer to Appendix 5.3.3 for notation related to the Malliavin calculus. Let us apply this result to $V = (R_T^{0,Y})^2$, such that we can write

$$\begin{aligned} &\mathbb{E}\left[(R_T^{0,Y})^2 \int_0^1 h^{(2)}(\eta Y_T + (1 - \eta)Y_T^0)(1 - \eta) d\eta\right] \\ &= \int_0^1 \mathbb{E}\left[h^{(2)}(\eta Y_T + (1 - \eta)Y_T^0)(R_T^{0,Y})^2\right](1 - \eta) d\eta \\ &= \int_0^1 \mathbb{E}\left[h^{(1)}(\eta Y_T + (1 - \eta)Y_T^0)V_1^\eta\right](1 - \eta) d\eta. \end{aligned} \quad (4.12)$$

Using now the L^p estimates in Lemmata 4.1 and B.1, we can show easily that

$$\|(R_T^{0,Y})^2\|_{1,2p} \leq_c (M_1 M_0 T)^2 \quad (4.13)$$

and get

$$\|V_1^\eta\|_p \leq_c \frac{(M_1 \sqrt{T})^2 (M_0 \sqrt{T})^2}{(1 - \rho^2) \lambda_{\inf} \sqrt{T}}. \quad (4.14)$$

Therefore, we can deduce that

$$\begin{aligned} & |\mathbb{E} \left[(R_T^{0,Y})^2 \int_0^1 h^{(2)}(\eta Y_T + (1 - \eta) Y_T^0) (1 - \eta) d\eta \right]| \\ & \leq_c \int_0^1 \|h^{(1)}(\eta Y_T + (1 - \eta) Y_T^0)\|_2 d\eta \frac{M_0}{\lambda_{\inf}(1 - \rho^2)} M_1 M_0^2 T^{\frac{3}{2}}. \end{aligned} \quad (4.15)$$

Because $\lambda_{\inf} \leq_c M_0$ and $\frac{1}{(1 - \rho^2)} \geq 1$, we finally obtain

$$|\text{Error}_2| \leq_c \left[\|h^{(1)}(Y_T^0)\|_2 + \int_0^1 \|h^{(1)}(\eta Y_T + (1 - \eta) Y_T^0)\|_2 d\eta \right] \frac{M_0^3 M_1}{\lambda_{\inf}(1 - \rho^2)} T^{\frac{3}{2}}. \quad (4.16)$$

Thus far, we have bounded the error using only $h^{(1)}$ for a smooth function h . In order to obtain a similar error bound under the assumption that h satisfies (\mathbb{S}_2) , we can use a density or regularization argument to approximate h by a sequence of smooth functions as in [Benhamou et al. \(2009, Section 5.2, Step 4\)](#).

4.2.2. Third order error analysis. We follow again the same strategy as for the second order case. By a Taylor expansion of Y_T around Y_T^0 , we have

$$Y_T = Y_T^0 + Y_{1,T} + \frac{1}{2} Y_{2,T} + \int_0^1 Y_{3,T}^\eta \frac{(1 - \eta)^2}{2} d\eta, \quad (4.17)$$

and by performing again a Taylor expansion for a smooth payoff h and taking expectations we obtain

$$\begin{aligned} \mathbb{E}[h(Y_T)] &= \mathbb{E}[h(Y_T^0)] + \mathbb{E}[h^{(1)}(Y_T^0)(Y_T - Y_T^0)] + \mathbb{E}\left[\frac{1}{2} h^{(2)}(Y_T^0)(Y_T - Y_T^0)^2\right] \\ &+ \mathbb{E}\left[(Y_T - Y_T^0)^3 \int_0^1 h^{(3)}(\eta Y_T + (1 - \eta) Y_T^0) \frac{(1 - \eta)^2}{2} d\eta\right]. \end{aligned} \quad (4.18)$$

Using (4.17), the latter becomes

$$\begin{aligned} \mathbb{E}[h(Y_T)] &= \mathbb{E}[h(Y_T^0)] + \mathbb{E}[h^{(1)}(Y_T^0) Y_{1,T}] + \mathbb{E}\left[h^{(1)}(Y_T^0) \frac{Y_{2,T}}{2}\right] \\ &+ \mathbb{E}\left[\frac{h^{(2)}(Y_T^0)}{2} Y_{1,T}^2\right] + \text{Error}_3, \end{aligned} \quad (4.19)$$

where

$$\text{Error}_3 = \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^1 Y_{3,T}^\eta \frac{(1-\eta)^2}{2} d\eta \right] \quad (4.20)$$

$$+ \mathbb{E} \left[(Y_T - Y_T^0)^3 \int_0^1 h^{(3)}(\eta Y_T + (1-\eta)Y_T^0) \frac{(1-\eta)^2}{2} d\eta \right] \quad (4.21)$$

$$+ \mathbb{E} \left[\frac{h^{(2)}(Y_T^0)}{2} [(Y_T - Y_T^0)^2 - Y_{1,T}^2] \right]. \quad (4.22)$$

Let us bound each term in the error separately. The first term (4.20), using (4.2) with $i = 2$ in Lemma 4.1 and the Cauchy–Schwarz inequality, is estimated by

$$\left| \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^1 Y_{3,T}^\eta \frac{(1-\eta)^2}{2} d\eta \right] \right| \leq_c \|h^{(1)}(Y_T^0)\|_2 M_1 M_0^3 T^2. \quad (4.23)$$

The second term (4.21) is handled as in the previous section. We recall that $Y_T - Y_T^0 = R_T^{0,Y} = \int_0^1 Y_{1,T}^\eta d\eta$ and apply Lemma B.2 with $k = 2$ to $V = (R_T^{0,Y})^3$ such that we can write

$$\begin{aligned} & \mathbb{E} \left[(R_T^{0,Y})^3 \int_0^1 h^{(3)}(\eta Y_T + (1-\eta)Y_T^0) \frac{(1-\eta)^2}{2} d\eta \right] \\ &= \int_0^1 \mathbb{E} \left[h^{(3)}(\eta Y_T + (1-\eta)Y_T^0) (R_T^{0,Y})^3 \right] \frac{(1-\eta)^2}{2} d\eta \\ &= \int_0^1 \mathbb{E} \left[h^{(1)}(\eta Y_T + (1-\eta)Y_T^0) V_2^\eta \right] \frac{(1-\eta)^2}{2} d\eta. \end{aligned} \quad (4.24)$$

Using the L^p estimates in Lemmata 4.1 and B.1, we show easily that

$$\|(R_T^{0,Y})^2\|_{2,2p} \leq_c (M_1 M_0 T)^3, \quad (4.25)$$

hence

$$\|V_2^\eta\|_p \leq_c \left(\frac{M_0}{\lambda_{\inf}(1-\rho^2)} \right)^2 M_1 M_0^3 T^2. \quad (4.26)$$

Therefore, we can deduce that

$$\begin{aligned} & \left| \mathbb{E} \left[(R_T^{0,Y})^3 \int_0^1 h^{(3)}(\eta Y_T + (1-\eta)Y_T^0) \frac{(1-\eta)^2}{2} d\eta \right] \right| \\ & \leq_c \int_0^1 \|h^{(1)}(\eta Y_T + (1-\eta)Y_T^0)\|_2 d\eta \frac{M_0^5 M_1}{(\lambda_{\inf}(1-\rho^2))^2} T^2. \end{aligned} \quad (4.27)$$

As for the third term (4.22), let us first provide a more explicit representation of $(Y_T - Y_T^0)^2 - Y_{1,T}^2$. We define

$$f(\eta) = (Y_T^\eta - Y_T^0)^2 \quad (4.28)$$

and perform a second order Taylor expansion around 0 to get

$$f(\eta) = f(0) + f^{(1)}(0)\eta + f^{(2)}(0)\frac{\eta^2}{2} + \int_0^\eta \frac{(\eta-t)^2}{2} f^{(3)}(t)dt \quad (4.29)$$

where

$$\begin{cases} f(0) = f^{(1)}(0) = 0 & f^{(1)}(\eta) = 2Y_{1,T}^\eta(Y_T^\eta - Y_T^0) \\ f^{(2)}(0) = 2Y_{1,T}^2 & f^{(2)}(\eta) = 2[Y_{2,T}^\eta(Y_T^\eta - Y_T^0) + (Y_{1,T}^\eta)^2] \\ f^{(3)}(0) = 6Y_{1,T}Y_{2,T} & f^{(3)}(\eta) = 2[Y_{3,T}^\eta(Y_T^\eta - Y_T^0) + 3Y_{1,T}^\eta Y_{2,T}^\eta]. \end{cases} \quad (4.30)$$

Setting $\eta = 1$ in (4.29), we obtain

$$(Y_T - Y_T^0)^2 = Y_{1,T}^2 + \int_0^1 (1-\eta)[Y_{3,T}^\eta(Y_T^\eta - Y_T^0) + 3Y_{1,T}^\eta Y_{2,T}^\eta]d\eta. \quad (4.31)$$

Replacing (4.31) into (4.22) and using Fubini's theorem, we get

$$\begin{aligned} & \mathbb{E} \left[\frac{h^{(2)}(Y_T^0)}{2} \int_0^1 (1-\eta)[Y_{3,T}^\eta(Y_T^\eta - Y_T^0) + 3Y_{1,T}^\eta Y_{2,T}^\eta]d\eta \right] \\ &= \int_0^1 \frac{(1-\eta)}{2} \mathbb{E} \left[h^{(2)}(Y_T^0)(Y_{3,T}^\eta(Y_T^\eta - Y_T^0) + 3Y_{1,T}^\eta Y_{2,T}^\eta)d\eta \right] \\ &= \int_0^1 \frac{(1-\eta)}{2} \mathbb{E} \left[h^{(1)}(Y_T^0)V_1^\eta d\eta \right], \end{aligned} \quad (4.32)$$

where for the last equality we have applied the integration-by-parts formula of Lemma B.2 with $V = Y_{3,T}^\eta(Y_T^\eta - Y_T^0) + 3Y_{1,T}^\eta Y_{2,T}^\eta$ for $k = 1$. Applying now the Cauchy–Schwartz inequality, we get the following error estimate

$$\left| \mathbb{E} \left[\frac{h^{(2)}(Y_T^0)}{2} [(Y_T - Y_T^0)^2 - Y_{1,T}^2] \right] \right| \leq c \|h^{(1)}(Y_T^0)\|_2 \int_0^1 \|V_1^\eta\|_2 d\eta, \quad (4.33)$$

while the L^p estimates in Lemmata 4.1 and B.1 yield, for $p \geq 1$, that

$$\|V\|_{1,2p} \leq c M_1^2 M_0^3 T^{\frac{5}{2}} \quad (4.34)$$

and

$$\|V_1^\eta\|_p \leq c \left(\frac{M_0}{\lambda_{\inf}(1-\rho^2)} \right) M_1 M_0^3 T^2. \quad (4.35)$$

Therefore, the third error term (4.33) is estimated by

$$\left| \mathbb{E} \left[\frac{h^{(2)}(Y_T^0)}{2} [(Y_T - Y_T^0)^2 - Y_{1,T}^2] \right] \right| \leq c \|h^{(1)}(Y_T^0)\|_2 \left(\frac{M_0}{\lambda_{\inf}(1-\rho^2)} \right) M_1 M_0^3 T^2. \quad (4.36)$$

Finally, using again that $\lambda_{\inf} \leq c M_0$ and $1 \leq \frac{1}{(1-\rho^2)}$, and by regrouping all the estimates in (4.23), (4.27) and (4.36), the third order error can be estimated as follows:

$$\begin{aligned} |\text{Error}_3| &\leq c \left[\|h^{(1)}(Y_T^0)\|_2 + \int_0^1 \|h^{(1)}(\eta Y_T + (1-\eta)Y_T^0)\|_2 d\eta \right] \\ &\quad \times \left(\frac{M_0}{\lambda_{\inf}(1-\rho^2)} \right)^2 M_1 M_0^3 T^2. \end{aligned} \quad (4.37)$$

4.3. Computation of the Greek coefficients. This subsection is devoted to the computation of the correction terms in the second order expansion of Theorem 3.6. The analogous derivation for the third order expansion is postponed to Appendix C. The correction terms are expressed in terms of Greeks of the payoff function around the proxy model, recall Definition 3.3, and we provide below a useful lemma for their computation.

Lemma 4.4. *Let θ be a continuous (or piecewise continuous) function and f be a function satisfying Assumption (\mathbb{S}_1) . Then it holds*

$$\mathbb{E} \left[\bar{f} \left(\int_0^T \lambda_t dW_t^L \right) \int_0^T \xi_t \theta_t dt \right] = \omega(\alpha, \theta)_0^T g_0^f(Y_T^0) + \omega(\lambda^2, \theta)_0^T g_1^f(Y_T^0), \quad (4.38)$$

$$\mathbb{E} \left[\bar{f} \left(\int_0^T \lambda_t dW_t^L \right) \int_0^T \gamma_t \theta_t dt \right] = \omega(\beta, \theta)_0^T g_0^f(Y_T^0) + \rho \omega(\sigma \lambda, \theta)_0^T g_1^f(Y_T^0), \quad (4.39)$$

$$\begin{aligned} \mathbb{E} \left[\bar{f} \left(\int_0^T \lambda_t dW_t^L \right) \int_0^T Y_{1,t} \theta_t dt \right] &= [\omega(\alpha, \alpha_y, \theta)_0^T + \omega(\beta, \alpha_z, \theta)_0^T] g_0^f(Y_T^0) \\ &\quad + [\omega(\lambda^2, \alpha_y, \theta)_0^T + \omega(\alpha, \lambda_y \lambda, \theta)_0^T] g_1^f(Y_T^0) \\ &\quad + \omega(\lambda^2, \lambda_y \lambda, \theta)_0^T g_2^f(Y_T^0) + \rho \omega(\sigma \lambda, \alpha_z, \theta)_0^T g_1^f(Y_T^0), \end{aligned} \quad (4.40)$$

$$\begin{aligned} \mathbb{E} \left[\bar{f} \left(\int_0^T \lambda_t dW_t^L \right) \int_0^T Z_{1,t} \theta_t dt \right] &= \omega(\beta, \beta_z, \theta)_0^T g_0^f(Y_T^0) \\ &\quad + \rho [\omega(\sigma \lambda, \beta_z, \theta)_0^T + \omega(\beta, \lambda \sigma_z, \theta)_0^T] g_1^f(Y_T^0) \\ &\quad + \rho^2 \omega(\sigma \lambda, \lambda \sigma_z, \theta)_0^T g_2^f(Y_T^0), \end{aligned} \quad (4.41)$$

$$\begin{aligned} \mathbb{E} \left[\bar{f} \left(\int_0^T \lambda_t dW_t^L \right) \int_0^T \xi_t^2 \theta_t dt \right] &= [\omega(\lambda^2, \theta)_0^T + 2\omega(\alpha, \alpha, \theta)_0^T] g_0^f(Y_T^0) \\ &\quad + 2[\omega(\lambda^2, \alpha, \theta)_0^T + \omega(\alpha, \lambda^2, \theta)_0^T] g_1^f(Y_T^0) \\ &\quad + 2\omega(\lambda^2, \lambda^2, \theta)_0^T g_2^f(Y_T^0), \end{aligned} \quad (4.42)$$

$$\begin{aligned} \mathbb{E} \left[\bar{f} \left(\int_0^T \lambda_t dW_t^L \right) \int_0^T \gamma_t^2 \theta_t dt \right] &= [\omega(\sigma^2, \theta)_0^T + 2\omega(\beta, \beta, \theta)_0^T] g_0^f(Y_T^0) \\ &\quad + 2\rho [\omega(\sigma \lambda, \beta, \theta)_0^T + \omega(\beta, \sigma \lambda, \theta)_0^T] g_1^f(Y_T^0) \\ &\quad + 2\rho^2 \omega(\sigma \lambda, \sigma \lambda, \theta)_0^T g_2^f(Y_T^0), \end{aligned} \quad (4.43)$$

$$\begin{aligned} \mathbb{E} \left[\bar{f} \left(\int_0^T \lambda_t dW_t^L \right) \int_0^T \xi_t \gamma_t \theta_t dt \right] &= [\omega(\alpha, \beta, \theta)_0^T + \omega(\beta, \alpha, \theta)_0^T] g_0^f(Y_T^0) \\ &\quad + [\omega(\lambda^2, \beta, \theta)_0^T + \omega(\beta, \lambda^2, \theta)_0^T] g_1^f(Y_T^0) \\ &\quad + \rho [\omega(\lambda \sigma, \theta)_0^T g_0^f(Y_T^0) + (\omega(\sigma \lambda, \alpha, \theta)_0^T + \omega(\alpha, \sigma \lambda, \theta)_0^T) g_1^f(Y_T^0) \\ &\quad + (\omega(\lambda^2, \sigma \lambda, \theta)_0^T + \omega(\sigma \lambda, \lambda^2, \theta)_0^T) g_2^f(Y_T^0)], \end{aligned} \quad (4.44)$$

where $\bar{f}(x) = f(y_0 + \int_0^T \alpha_t dt + x)$, while the processes ξ and γ are defined in (4.47) and (4.48) respectively.

Proof. The equalities are derived by laborious calculations using Itô's formula and by successively applying Lemmata 4.2 and 4.3. The details are omitted for the sake of brevity. \square

4.3.1. *Greek coefficients for the second order approximation.* The correction term for the second order expansion is provided by $\mathbb{E}[h^{(1)}(Y_T^0)Y_{1,T}]$ in (4.8) and our target now is to make this explicit. Let us recall equations (3.15)–(3.19), that $Y_{1,t} = \frac{\partial Y_t^\eta}{\partial \eta}|_{\eta=0}$ and $Z_{1,t} = \frac{\partial Z_t^\eta}{\partial \eta}|_{\eta=0}$ and Remark 3.5, which together yield that

$$Y_{1,T} = \int_0^T \xi_t \alpha_{y,t} dt + \int_0^T \gamma_t \alpha_{z,t} dt + \int_0^T \xi_t \lambda_{y,t} dW_t^L, \quad (4.45)$$

$$Z_{1,T} = \int_0^T \gamma_t \beta_{z,t} dt + \int_0^T \gamma_t \sigma_{z,t} dW_t^X, \quad (4.46)$$

$$\xi_t = Y_t^0 - y_0 = \int_0^t \alpha_s ds + \int_0^t \lambda_s dW_t^L, \quad (4.47)$$

$$\gamma_t = Z_t^0 - z_0 = \int_0^t \beta_s ds + \int_0^t \sigma_s dW_t^X. \quad (4.48)$$

Let us also introduce the shifted payoff function

$$\bar{h}^{(i)}(x) = h^{(i)}\left(y_0 + \int_0^T \alpha_t dt + x\right), \quad \text{for } i \in \mathbb{N}. \quad (4.49)$$

Then we have that

$$\begin{aligned} \mathbb{E}\left[h^{(1)}(Y_T^0)Y_{1,T}\right] &= \mathbb{E}\left[\bar{h}^{(1)}\left(\int_0^T \lambda_t dW_t^L\right) \int_0^T \xi_t \alpha_{y,t} dt\right] \\ &+ \mathbb{E}\left[\bar{h}^{(1)}\left(\int_0^T \lambda_t dW_t^L\right) \int_0^T \gamma_t \alpha_{z,t} dt\right] + \mathbb{E}\left[\bar{h}^{(1)}\left(\int_0^T \lambda_t dW_t^L\right) \int_0^T \xi_t \lambda_{y,t} dW_t^L\right]. \end{aligned} \quad (4.50)$$

By applying Lemmata 4.3 and 4.4, we obtain

$$\mathbb{E}\left[\bar{h}^{(1)}\left(\int_0^T \lambda_t dW_t^L\right) \int_0^T \xi_t \alpha_{y,t} dt\right] = \omega(\alpha, \alpha_y)_0^T g_1^h(Y_T^0) + \omega(\lambda^2, \alpha_y)_0^T g_2^h(Y_T^0), \quad (4.51)$$

$$\mathbb{E}\left[\bar{h}^{(1)}\left(\int_0^T \lambda_t dW_t^L\right) \int_0^T \gamma_t \alpha_{z,t} dt\right] = \omega(\beta, \alpha_z)_0^T g_1^h(Y_T^0) + \rho\omega(\sigma\lambda, \alpha_z)_0^T g_2^h(Y_T^0), \quad (4.52)$$

$$\mathbb{E}\left[\bar{h}^{(1)}\left(\int_0^T \lambda_t dW_t^L\right) \int_0^T \xi_t \lambda_{y,t} dW_t^L\right] = \omega(\alpha, \lambda_y \lambda)_0^T g_2^h(Y_T^0) + \omega(\lambda^2, \lambda_y \lambda)_0^T g_3^h(Y_T^0). \quad (4.53)$$

More specifically, the first equality follows directly by (4.38) and the second one by (4.39). For the third equality, we apply first Lemma 4.3 and then (4.38).

Finally, by gathering all the terms, passing to the initial parameters via (4.54) below, and writing them as a second order polynomial in ρ , we arrive at (3.27).

$$\begin{cases} \omega(\alpha, \alpha_y)_0^T &= \frac{1}{2}\omega(\lambda^2, \lambda_y\lambda)_0^T + \rho \left[\frac{1}{2}\omega(\lambda^2, \lambda_y\sigma)_0^T + \omega(\lambda\sigma, \lambda_y\lambda)_0^T \right] + \rho^2\omega(\lambda\sigma, \lambda_y\sigma)_0^T, \\ \omega(\beta, \alpha_z)_0^T &= \frac{1}{2}\rho\omega(\sigma^2, \lambda\sigma_z)_0^T, \\ \omega(\lambda^2, \alpha_y)_0^T &= -\omega(\lambda^2, \lambda_y\lambda)_0^T - \rho\omega(\lambda^2, \lambda_y\sigma)_0^T, \\ \omega(\alpha, \lambda_y\lambda)_0^T &= -\frac{1}{2}\omega(\lambda^2, \lambda_y\lambda)_0^T - \rho\omega(\lambda\sigma, \lambda_y\lambda)_0^T, \\ \omega(\lambda\sigma, \alpha_z)_0^T &= -\rho\omega(\sigma\lambda, \lambda\sigma_z)_0^T. \end{cases} \quad (4.54)$$

5. NUMERICAL EXPERIMENTS

This section is dedicated to numerical experiments and a comparison of the second and third order expansions with the “market” approximation for quanto options.

5.1. Time-homogeneous hyperbolic local volatility model. We consider the time-homogeneous hyperbolic local volatility model where the SDEs for the forward LIBOR and the forward FX rate are provided by (2.2) and (2.5), while the coefficients $\lambda(\cdot, y)$ and $\sigma(\cdot, z)$ are homogeneous in time and take the form:

$$\lambda(y) := \nu_L \left[\frac{1 - \beta_L + \beta_L^2}{\beta_L} + \frac{(\beta_L - 1)}{\beta_L} \left(\frac{\sqrt{y^2 + \beta_L^2(1 - y)^2} - \beta_L}{y} \right) \right], \quad (5.1)$$

$$\sigma(z) := \nu_X \left[\frac{1 - \beta_X + \beta_X^2}{\beta_X} + \frac{(\beta_X - 1)}{\beta_X} \left(\frac{\sqrt{z^2 + \beta_X^2(1 - z)^2} - \beta_X}{z} \right) \right], \quad (5.2)$$

where ν_L and ν_X , both strictly positive, represent the levels of volatility, while β_L and β_X , both valued in $[0, 1]$, represent the skew parameters. This model corresponds to the Black–Scholes model for $\beta_L = \beta_X = 1$ and exhibits a skew for the implied volatility surface when β_L or $\beta_X \neq 1$. It was introduced by Jäckel (2008), behaves similarly to the CEV (Constant Elasticity of Variance) model, and has been used for numerical experiments also in Bompis and Hok (2014). The advantage of this model is that zero is not an attainable boundary, and that allows to avoid some numerical instabilities present in the CEV model when the underlying LIBOR or FX rate are close to zero; see *e.g.* Andreasen and Andersen (2000). Although the assumptions of boundedness and ellipticity are not fulfilled, we reasonably expect that our approximation formulas remain valid for this model, and apply Theorems 3.6 and 3.7. The payoff of a call option on the other hand does satisfy the smoothness assumption (\mathbb{S}_2) , as the payoff is everywhere differentiable apart from the kink at the strike level and grows exponentially. The numerical experiments that follow show that the derived approximations perform well in this setting, even though some theoretical assumptions are not satisfied.

5.2. Market approximation for pricing of European quanto option. The common market practice is to evaluate European quanto call/put options analytically using a Black–Scholes type formula with a quanto drift correction. More precisely, for a caplet with maturity date T , strike K and payment date T_1 , the market approximation is provided by

$$\mathbb{C}^M(T, K) = \delta B(0, T_1) \left(e^{y_0 - qT} \Phi(d_1) - e^k \Phi(d_2) \right), \quad (5.3)$$

where $q = \rho\lambda_{\text{imp}}(T, \text{ATM})\sigma_{\text{imp}}(T, \text{ATM})$, $k = \ln K$, Φ is the cdf of the standard normal distribution, and

$$d_1 = \frac{y_0 - k - \rho\lambda_{\text{imp}}(T, \text{ATM})\sigma_{\text{imp}}(T, \text{ATM})T + \frac{1}{2}\lambda_{\text{imp}}(T, k)^2T}{\lambda_{\text{imp}}(T, k)\sqrt{T}}, \quad (5.4)$$

$$d_2 = d_1 - \lambda_{\text{imp}}(T, k)\sqrt{T}, \quad (5.5)$$

where $\lambda_{\text{imp}}(T, \text{ATM})$, $\sigma_{\text{imp}}(T, \text{ATM})$ are respectively the ATM implied volatility for the forward LIBOR rate and the FX forward rate with expiry T , while $\lambda_{\text{imp}}(T, k)$ is implied volatility for the

forward LIBOR rate with strike k . This approach is similar to the “practitioner” Black–Scholes model considered in [Christoffersen and Jacobs \(2004\)](#) or [Romo \(2012\)](#). Observe that the approximation formula (5.3) becomes exact by construction when $\rho = 0$.

5.3. Comparison results for the second and third order expansions and the market approximation.

5.3.1. *Set of parameters.* The numerical experiments are conducted using the following values for the parameters: $L_0 = 6\%$, $X_0 = 1$, $\nu_L = 8\%$, $\beta_L = 0.3$, $\nu_X = 15\%$ and $\beta_X = 0.5$. They are chosen to be comparable to market values, see *e.g.* [Hull and White \(2000\)](#) and [Ng and Sun \(2008\)](#). In order to illustrate this, Figures 5.1 and 5.2 show, respectively, the implied volatilities for the forward LIBOR and the forward FX rates generated with these parameters for various maturities. They represent the skew typically observed in interest rates and FX markets.

The challenging part for the pricing comes from the choice of the correlation parameter ρ between the forward LIBOR rate and the foreign exchange rate, because its level is not directly observable in the market and has a significant impact in the pricing as showed in Figure 5.3. In practice, its level is either chosen by the trader or estimated using historical data. The empirical analysis in [Boenkost and Schmidt \(2003\)](#) shows that the estimated correlations depend on the underlying interest rates and the pair of currencies considered. In general, ρ is not too *large* and belongs to the region $[-0.2, 0.2]$. A trader who sells this product, may choose its level in a conservative way (higher selling price) by taking a lower or negative correlation level, as the option price is decreasing with ρ . For these reasons and for the purpose of testing our formulas, we consider correlation levels $\rho \in [-0.5, 0.5]$.

In order for the tests to be comprehensive, we consider various relevant maturities (1, 6, 10 and 15 years) and strikes (with a range increasing with the maturity). This roughly covers very out-of-the-money options and very in-the-money options.

5.3.2. *Benchmarks.* Benchmarks for model prices are computed using the Monte Carlo method by discretizing the diffusion process using the Euler scheme. The number of Monte Carlo (MC) paths and the number of steps in the discretization are chosen such that the 95% confidence intervals are within 2 basis points.

5.3.3. *Accuracy.* The results for the tests are illustrated in Figures 5.4, 5.5, 5.6 and 5.7. The following observations stem from these tests and their illustrations:

In general, the test results up to 15 years show that the second and third order approximation formulas provide very good accuracy. Table 5.1 gives some statistics (average and maximum for the absolute discrepancy) for various correlation values considered. The maximum average error for the second (third) order approximation formulas is 2.8 (2.2) bps with correlation value equal to -0.5 . The maximum error for the second (third) order approximation is about 14 (8.4) bps. Third order approximation formulas produce better accuracy in comparison to the second order approximation formulas, which is expected.

The market approximation formulas provide good accuracy as well; see the statistics in Table 5.2. This is due to the fact that this formula is exact for $\rho = 0$, which results in good accuracy when the correlation parameter is fairly small. Indeed, the largest average error for the market approximation formulas is 3.3 bps with correlation value equal to -0.5 . The maximum error for the market approximation is about 12.6 bps.

In order to compare the different methods, let us mention that when the impact of the quanto effect becomes important (*i.e.* $\rho = \pm 0.5$), the accuracy of the third order approximation is better than the one given by the market approximation (see Figures 5.4 and 5.7), whereas the precision given by the second order and the market approximations is comparable. For a reduced quanto effect (*i.e.* $\rho = \pm 0.2$), the accuracy from the third order and the market approximation is similar. Indeed, in the limiting case of $\rho \rightarrow 0$, the market approximation becomes exact by construction. The main advantage of our expansion formulas is to provide an accurate estimation of the error

which is directly related to the maturity of the option (T), the level and curvature of the local volatility functions (M_0 and M_1) and the quanto impact (ρ).

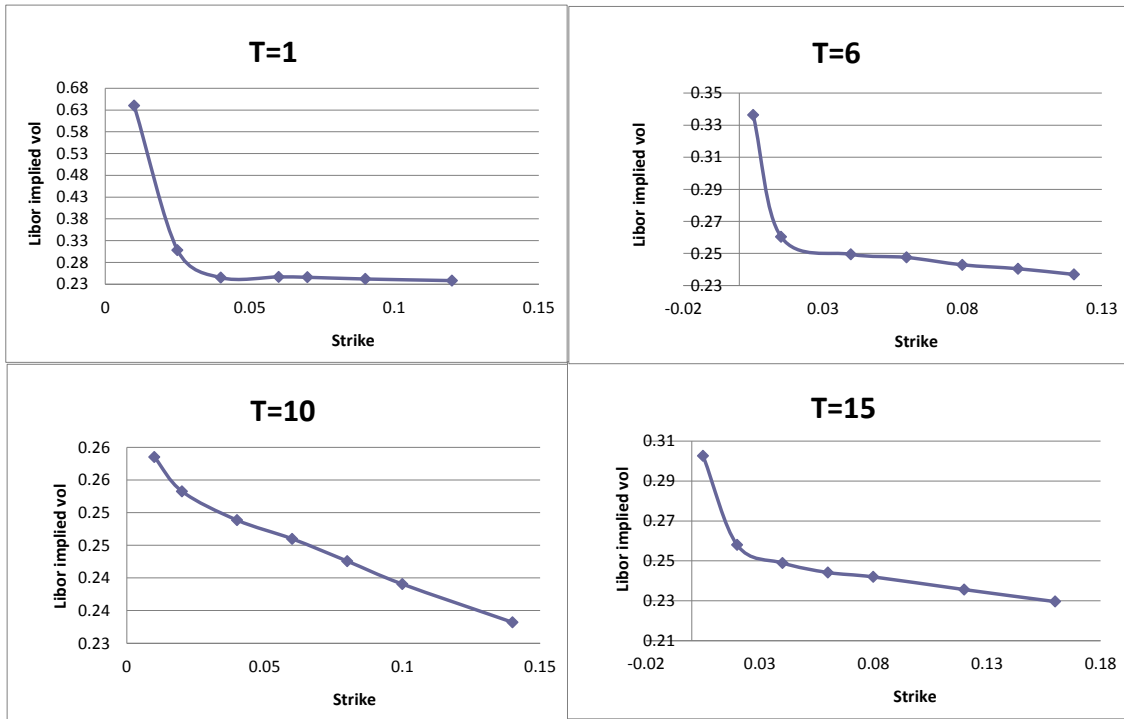


FIGURE 5.1. Forward LIBOR rate implied volatility generated with parameters $L_0 = 6\%$, $\nu_L = 8\%$, $\beta_L = 0.3$ for various maturities.

Correlation	Average (2nd order)	MAX (2nd order)	Average (3rd order)	MAX(3rd order)
-0.5	0.00028	0.00141	0.00022	0.00084
-0.2	0.00014	0.00074	0.00006	0.0002
0.2	0.00007	0.00041	0.00004	0.00017
0.5	0.00007	0.00035	0.00003	0.00011

TABLE 5.1. Average and maximum statistics for the absolute discrepancy, for various correlation values considered

Correlation	Average (market approximation)	MAX (market approximation)
-0.5	0.00033	0.00126
-0.2	0.00005	0.00018
0.2	0.00006	0.00027
0.5	0.00023	0.00087

TABLE 5.2. Average and maximum statistics for the absolute discrepancy, for various correlation values considered

APPENDIX A. PROOF OF LEMMA 3.9

Proof. Let us write

$$\mathbb{C}^{BS}(y_0) = e^{-\Sigma(T)} \tilde{\mathbb{C}}^{BS}(y_0) \quad (\text{A.1})$$

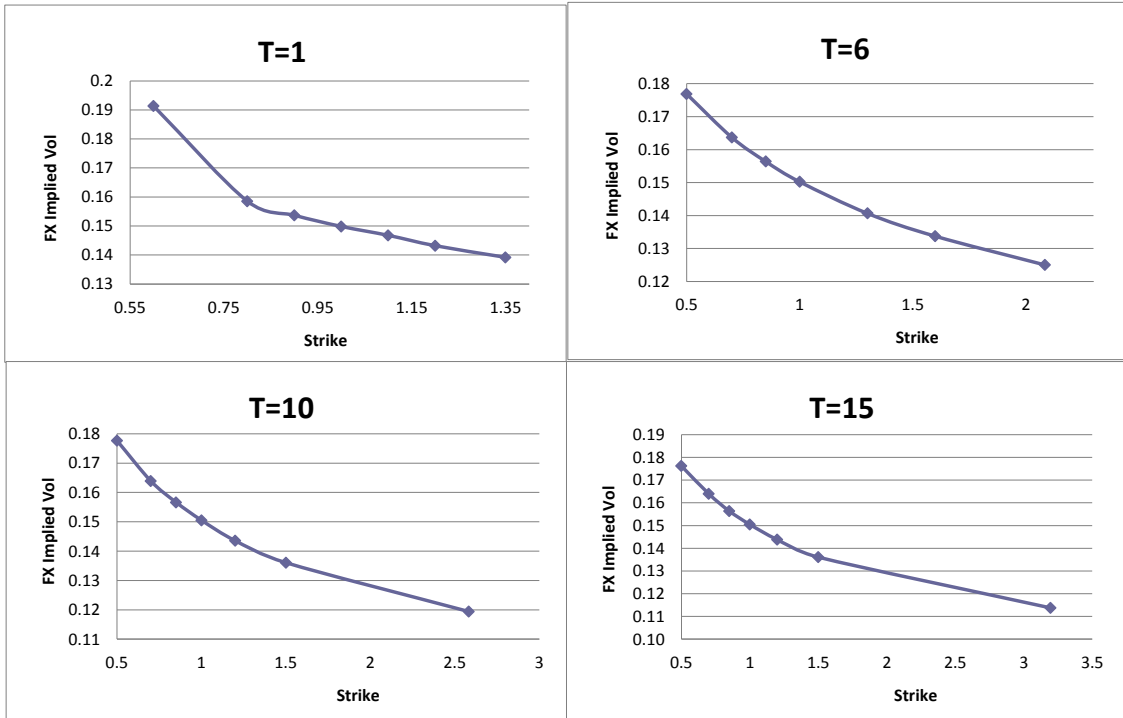


FIGURE 5.2. FX forward rate implied volatility generated with parameters $X_0 = 1$, $\nu_X = 15\%$, $\beta_X = 0.5$ for various maturities.

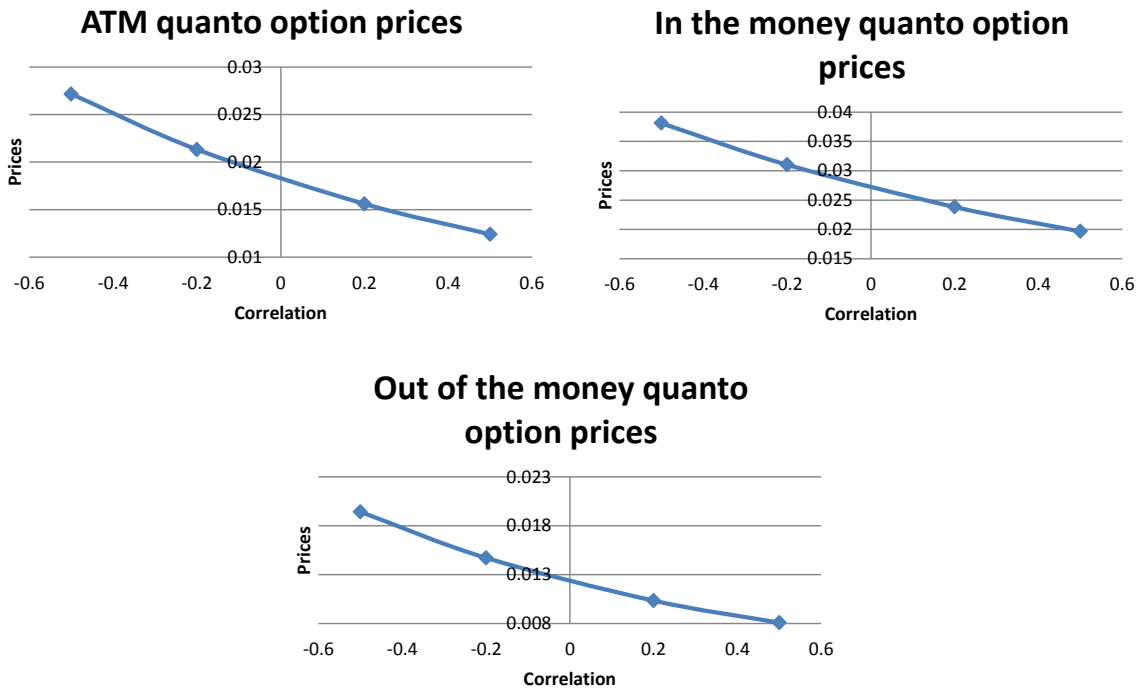


FIGURE 5.3. Impact of the correlation parameter ρ for in-the-money ($K = 4\%$), ATM ($K = 6\%$) and out-of-the-money ($K = 8\%$) option prices.

where

$$\tilde{C}^{BS}(y_0) = e^{y_0} \Phi(d_1) - e^{\tilde{k}} \Phi(d_2) \tag{A.2}$$

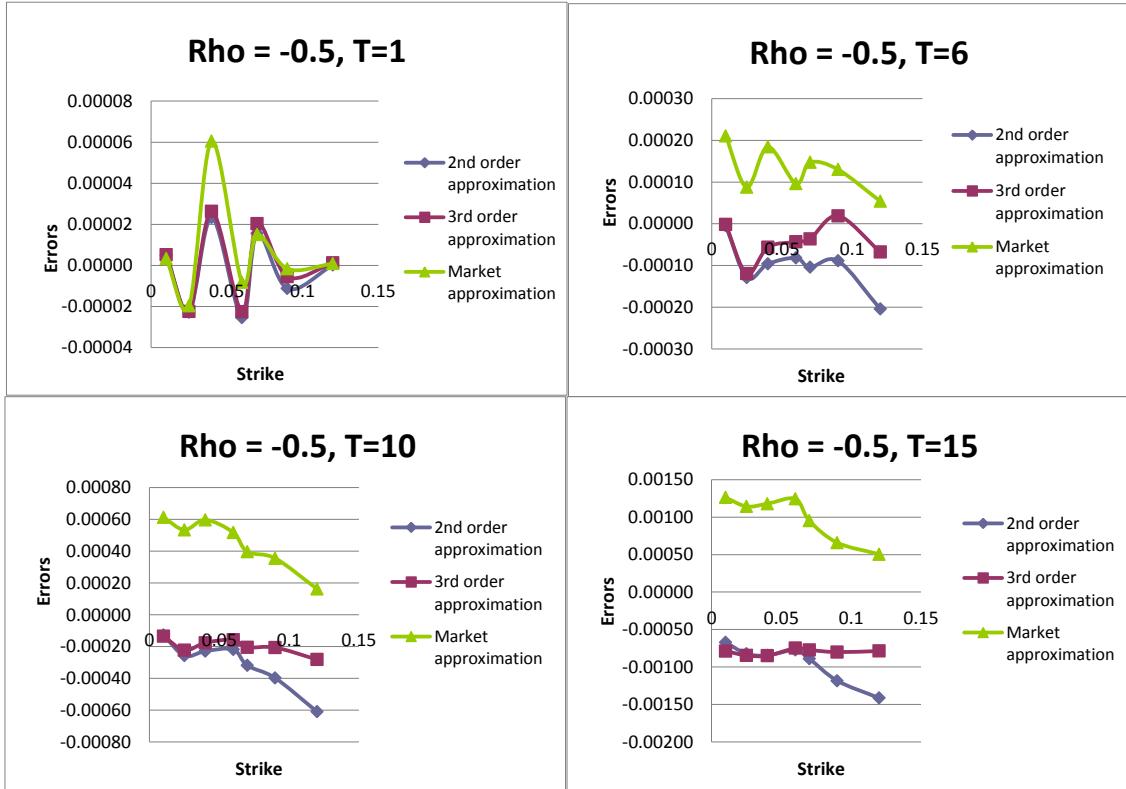


FIGURE 5.4. Absolute discrepancy between the benchmarks prices and those calculated with different approximation schemes when $\rho = -0.5$.

with

$$\tilde{k} = k + \Sigma(T), \quad d_1 = \frac{y_0 - \tilde{k} + \frac{1}{2}\Lambda(T)}{\sqrt{\Lambda(T)}}, \quad d_2 = d_1 - \sqrt{\Lambda(T)}. \quad (\text{A.3})$$

For $n = 1$, we get $\frac{\partial}{\partial y} \tilde{C}^{BS}(y_0) = e^{y_0} \Phi(d_1)$. For $n \geq 2$, we apply the Leibniz formula for the product $e^{y_0} \Phi(d_1)$. \square

APPENDIX B. MALLIAVIN CALCULUS

We start by introducing some definitions and notation for the Malliavin calculus — see *e.g.* [Bally, Caramellino, and Lombardi \(2010\)](#) or [Nualart \(2005\)](#) more for details — before providing two lemmas for the L^p estimates of Malliavin derivatives and the integration-by-parts formulas.

Let us write $W^L = \rho W^X + \sqrt{1 - \rho^2} \tilde{W}^L$, where $(\tilde{W}_t^L)_{0 \leq t \leq T}$ is a Brownian motion independent of $(W_t^X)_{0 \leq t \leq T}$, and consider the Malliavin calculus for the 2-dimensional Brownian motion (\tilde{W}^L, W^X) . Let $D_t^i F$, $i = 1, 2$, denote the Malliavin derivative of the random variable F wrt to the Brownian motion i at time t , and similarly for the higher order derivatives, where for example $D_{t_1, t_2}^{i, j} F = D_{t_1}^i D_{t_2}^j F$.

Under the regularity assumption (\mathbb{R}_4) , using [Nualart \(2005\)](#), we know that for any $t \leq T$, any $\eta \in [0, 1]$ and any $p \geq 1$, we have $(Y_t^\eta, Z_t^\eta) \in \mathbb{D}^{4, p}$, $(Y_{1, t}^\eta, Z_{1, t}^\eta) \in \mathbb{D}^{3, p}$, $(Y_{2, t}^\eta, Z_{2, t}^\eta) \in \mathbb{D}^{2, p}$ and $(Y_{3, t}^\eta, Z_{3, t}^\eta) \in \mathbb{D}^{1, p}$. The existence of any moment is easy to establish, see *e.g.* [Priouret \(2005\)](#) or [Nualart \(2005\)](#). We focus on the Malliavin differentiability of the system of SDEs and their L^p estimates.

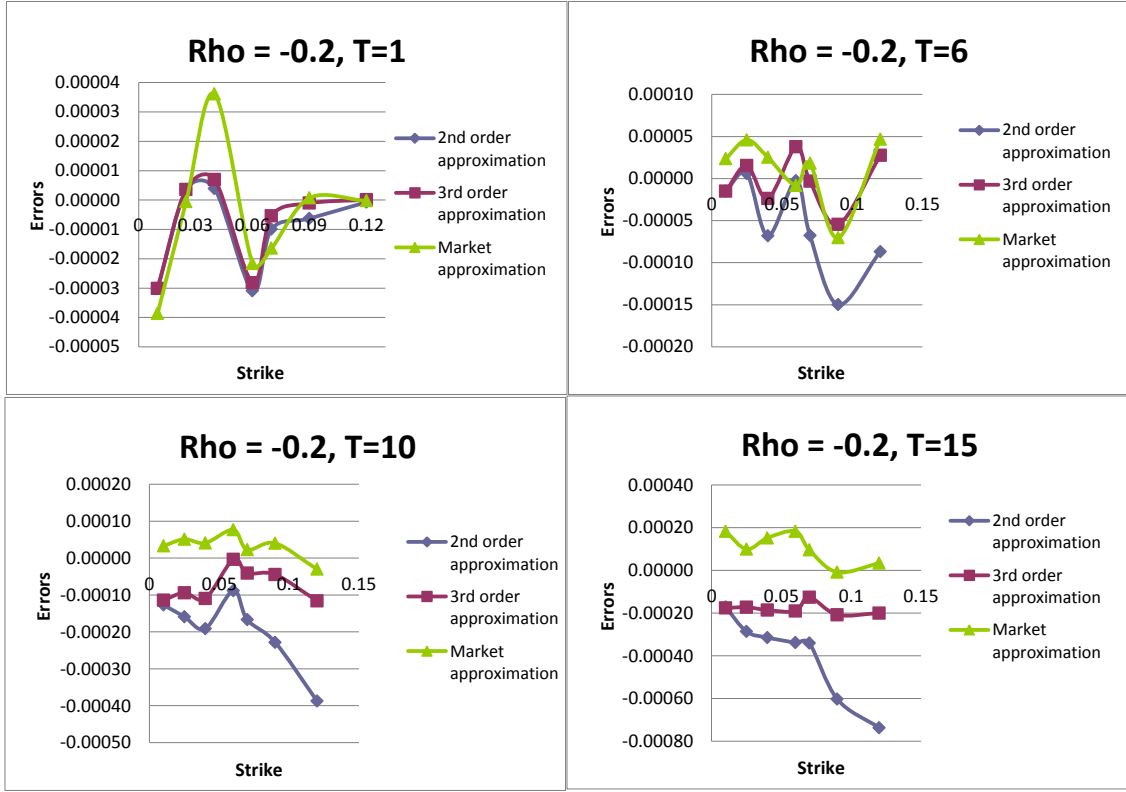


FIGURE 5.5. Absolute discrepancy between the benchmarks prices and those calculated with different approximation schemes when $\rho = -0.2$.

Let $r > t$, then $D_r^1 Y_t^\eta = D_r^2 Y_t^\eta = 0$. Now take $r \leq t$, then $(D_r^1 Y_t^\eta, D_r^2 Y_t^\eta)$ solves the following system of SDEs

$$\begin{cases} D_r^1 Y_t^\eta = \lambda(r, \eta Y_r^\eta + (1-\eta)y_0) \sqrt{1-\rho^2} + \int_r^t \eta \alpha_y D_r^1 Y_u^\eta du \\ \quad + \int_r^t \eta \lambda_y D_r^1 Y_u^\eta (\sqrt{1-\rho^2} d\widetilde{W}_t^L + \rho dW_t^X), \\ D_r^2 Y_t^\eta = \lambda(r, \eta Y_r^\eta + (1-\eta)y_0) \rho + \int_r^t \eta (\alpha_y D_r^2 Y_u^\eta + \alpha_z D_r^2 Z_u^\eta) du \\ \quad + \int_r^t \eta \lambda_y D_r^2 Y_u^\eta (\sqrt{1-\rho^2} d\widetilde{W}_t^L + \rho dW_t^X). \end{cases} \quad (\text{B.1})$$

For the second order Malliavin derivatives, where for instance $r < s \leq t$, we have

$$\begin{cases} D_{r,s}^{1,1} Y_t^\eta = \eta \lambda_y D_r^1 Y_s^\eta + \int_s^t \eta (\alpha_y D_{s,r}^{1,1} Y_u^\eta + \alpha_{yy} D_r^1 Y_u^\eta D_s^1 Y_u^\eta) du \\ \quad + \int_s^t \eta (\lambda_y D_{s,r}^{1,1} Y_u^\eta + \lambda_{yy} D_r^1 Y_u^\eta D_s^1 Y_u^\eta) (\sqrt{1-\rho^2} d\widetilde{W}_t^L + \rho dW_t^X), \\ D_{r,s}^{1,2} Y_t^\eta = \eta \lambda_y D_r^1 Y_s^\eta + \int_s^t \eta (\alpha_y D_{r,s}^{1,2} Y_u^\eta + \alpha_{yy} D_r^1 Y_u^\eta D_s^2 Y_u^\eta + \alpha_{yz} D_r^1 Y_u^\eta D_s^2 Z_u^\eta) du \\ \quad + \int_s^t \eta (\lambda_y D_{r,s}^{1,2} Y_u^\eta + \lambda_{yy} D_r^1 Y_u^\eta D_s^2 Y_u^\eta) (\sqrt{1-\rho^2} d\widetilde{W}_t^L + \rho dW_t^X), \\ D_{r,s}^{2,2} Y_t^\eta = \eta \lambda_y D_r^2 Y_s^\eta + \int_s^t \eta (\alpha_y D_{s,r}^{2,2} Y_u^\eta + \alpha_{yy} D_r^2 Y_u^\eta D_s^2 Y_u^\eta + \alpha_{yz} D_r^2 Y_u^\eta D_s^2 Z_u^\eta \\ \quad + \alpha_z D_{s,r}^{2,2} Z_u^\eta + \alpha_{zy} D_r^2 Z_u^\eta D_s^2 Y_u^\eta + \alpha_{zz} D_r^2 Z_u^\eta D_s^2 Z_u^\eta) du \\ \quad + \int_s^t \eta (\lambda_y D_{r,s}^{2,2} Y_u^\eta + \lambda_{yy} D_r^2 Y_u^\eta D_s^2 Y_u^\eta) (\sqrt{1-\rho^2} d\widetilde{W}_t^L + \rho dW_t^X). \end{cases} \quad (\text{B.2})$$

The process Z^η is independent from \widetilde{W}^L , hence its Malliavin derivatives wrt to it are zero. Similarly, for $r > t$, $D_r^2 Z_t^\eta = 0$, while for $r \leq t$, $D_r^2 Z_t^\eta$ solves

$$D_r^2 Z_t^\eta = \sigma(r, \eta Z_r^\eta + (1-\eta)z_0) + \int_r^t \eta \beta_z D_r^2 Z_u^\eta du + \int_r^t \eta \sigma_z D_r^2 Z_u^\eta dW_u^X. \quad (\text{B.3})$$

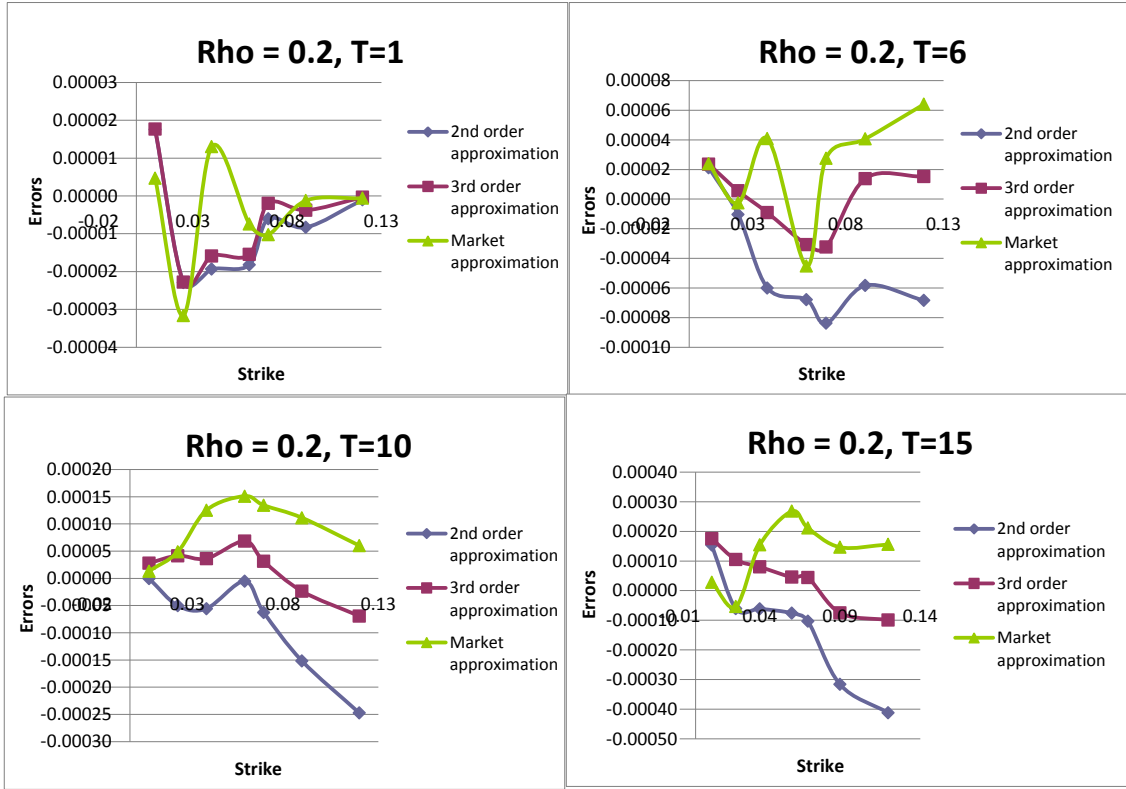


FIGURE 5.6. Absolute discrepancy between the benchmarks prices and those calculated with different approximation schemes when $\rho = 0.2$.

For the second order Malliavin derivatives, take for instance $r < s \leq t$, we have

$$\begin{cases} D_{r,s}^2 Z_t^\eta = \eta \sigma_z D_r^2 Z_s^\eta + \int_s^t \eta (\beta_{zz} D_s^2 Z_u^\eta D_r^2 Z_u^\eta + \beta_z D_{r,s}^2 Z_u^\eta) du \\ \quad + \int_s^t \eta (\sigma_{zz} D_s^2 Z_u^\eta D_r^2 Z_u^\eta + \sigma_z D_{r,s}^2 Z_u^\eta) dW_u^X. \end{cases} \quad (\text{B.4})$$

Similarly, we provide the Malliavin derivatives for $(Y_{1,t}^\eta, Z_{1,t}^\eta)$. $D^1 Z_{1,t}^\eta = 0$ because Z^η is independent of \tilde{W}^L . $D_r^2 Z_{1,t}^\eta = 0$ for $r > t$. For $r \leq t$ we have

$$\begin{cases} D_r^2 Z_{1,t}^\eta = \sigma_z (\eta Z_{1,r}^\eta + Z_r^\eta - z_0) \\ \quad + \int_r^t [\eta \beta_{zz} D_r^2 Z_u^\eta (\eta Z_{1,u}^\eta + Z_u^\eta - z_0) + \beta_z (\eta D_r^2 Z_{1,u}^\eta + D_r^2 Z_u^\eta)] du \\ \quad + \int_r^t [\eta \sigma_{zz} D_r^2 Z_u^\eta (\eta Z_{1,u}^\eta + Z_u^\eta - z_0) + \sigma_z (\eta D_r^2 Z_{1,u}^\eta + D_r^2 Z_u^\eta)] dW_u^X \end{cases} \quad (\text{B.5})$$

Furthermore, $D_r^i Y_{1,t}^\eta = 0$ for $r > t$, $i = 1, 2$, while for $r \leq t$, we get

$$\begin{cases} D_r^1 Y_{1,t}^\eta = \lambda_y (\eta Y_{1,r}^\eta + Y_r^\eta - y_0) + \int_r^t [\eta \alpha_{yy} D_r^1 Y_u^\eta (\eta Y_{1,u}^\eta + Y_u^\eta - y_0) \\ \quad + \alpha_y (\eta D_r^1 Y_{1,u}^\eta + D_r^1 Y_u^\eta) + \eta \alpha_{zy} D_r^1 Y_u^\eta (\eta Z_{1,u}^\eta + Z_u^\eta - z_0)] du \\ \quad + \int_r^t [\eta \lambda_{yy} D_r^1 Y_u^\eta (\eta Y_{1,u}^\eta + Y_u^\eta - y_0) + \lambda_y (\eta D_r^1 Y_{1,u}^\eta + D_r^1 Y_u^\eta)] \\ \quad (\sqrt{1 - \rho^2} d\tilde{W}_t^L + \rho dW_t^X), \\ D_r^2 Y_{1,t}^\eta = \rho \lambda_y (\eta Y_{1,r}^\eta + Y_r^\eta - y_0) \\ \quad + \int_r^t [\eta (\eta Y_{1,u}^\eta + Y_u^\eta - y_0) (\alpha_{yy} D_r^2 Y_u^\eta + \alpha_{yz} D_r^2 Z_u^\eta) + \alpha_y (\eta D_r^2 Y_{1,u}^\eta + D_r^2 Y_u^\eta) \\ \quad \eta (\eta Z_{1,u}^\eta + Z_u^\eta - z_0) (\alpha_{yz} D_r^2 Y_u^\eta + \alpha_{zz} D_r^2 Z_u^\eta) + \alpha_z (\eta D_r^2 Z_{1,u}^\eta + D_r^2 Z_u^\eta)] du \\ \quad + \int_r^t [\eta \lambda_{yy} D_r^2 Y_u^\eta (\eta Y_{1,u}^\eta + Y_u^\eta - y_0) + \lambda_y (\eta D_r^2 Y_{1,u}^\eta + D_r^2 Y_u^\eta)] \\ \quad (\sqrt{1 - \rho^2} d\tilde{W}_t^L + \rho dW_t^X). \end{cases} \quad (\text{B.6})$$

Other Malliavin derivatives for this system of SDEs can be derived similarly, without particular difficulties. Following [Benhamou et al. \(2009, Theorem 5.2, Step 2\)](#), we provide in the following

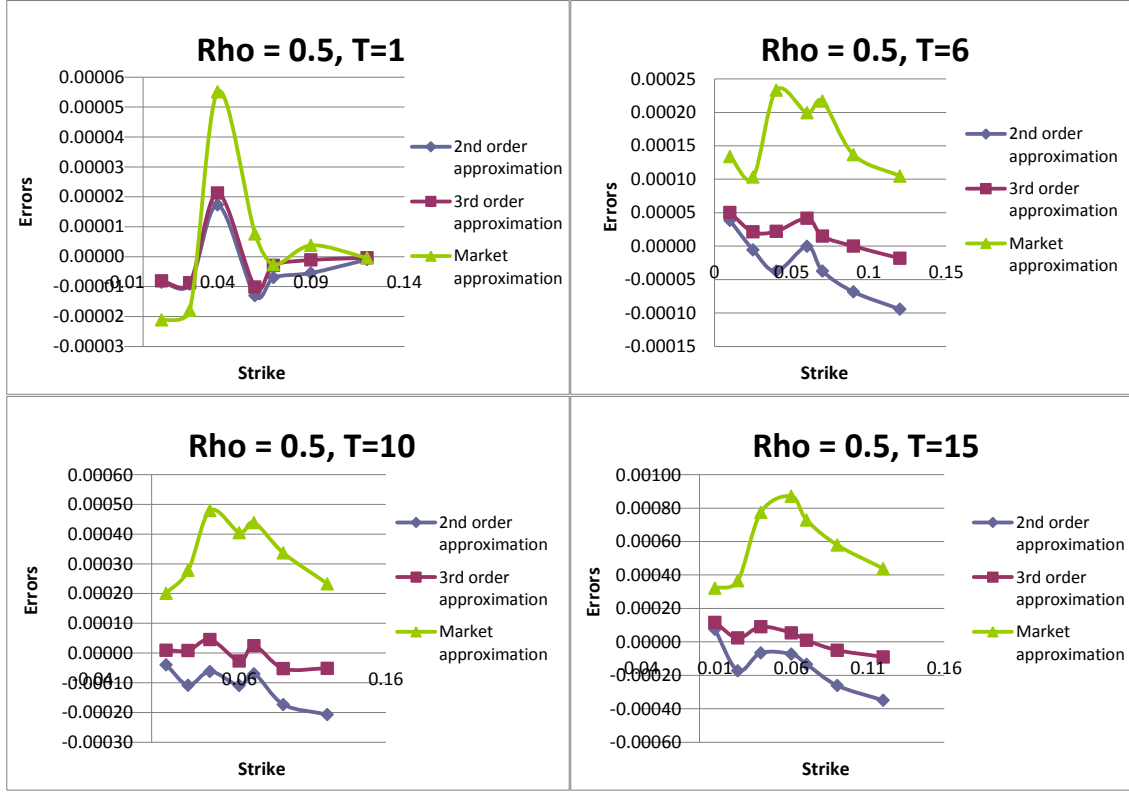


FIGURE 5.7. Absolute discrepancy between the benchmarks prices and those calculated with different approximation schemes when $\rho = 0.5$.

lemma tight estimates for this system of Malliavin derivatives, which are useful for the error analysis.

Lemma B.1 (Estimates of Malliavin derivatives). *The following hold, for any $p \geq 1$ and $i, j, k = 1, 2$:*

$$\begin{aligned}
\mathbb{E}|D_r^i Z_t^\eta|^p &\leq c |\sigma|_\infty^p & \mathbb{E}|D_r^i Y_t^\eta|^p &\leq c |\lambda|_\infty^p \\
\mathbb{E}|D_{r,s}^{i,j} Z_t^\eta|^p &\leq c |\sigma|_\infty^p M_1^p & \mathbb{E}|D_{r,s}^{i,j} Y_t^\eta|^p &\leq c |\lambda|_\infty^p M_1^p \\
\mathbb{E}|D_{r,s,u}^{i,j,k} Z_t^\eta|^p &\leq c |\sigma|_\infty^p M_0^p M_1^p & \mathbb{E}|D_{r,s,u}^{i,j,k} Y_t^\eta|^p &\leq c |\lambda|_\infty^p M_0^p M_1^p \\
\mathbb{E}|D_r^i Z_{1,t}^\eta|^p &\leq c M_1^p (M_0 \sqrt{T})^p & \mathbb{E}|D_r^i Y_{1,t}^\eta|^p &\leq c M_1^p (M_0 \sqrt{T})^p \\
\mathbb{E}|D_{r,s}^{i,j} Y_{1,t}^\eta|^p &\leq c M_0^p M_1^p & \mathbb{E}|D_{r,s}^{i,j} Z_{1,t}^\eta|^p &\leq c M_0^p M_1^p \\
\mathbb{E}|D_r^i Z_{2,t}^\eta|^p &\leq c M_1^p (M_0 \sqrt{T})^{2p} & \mathbb{E}|D_r^i Y_{2,t}^\eta|^p &\leq c M_1^p (M_0 \sqrt{T})^{2p} \\
\mathbb{E}|D_r^i Z_{3,t}^\eta|^p &\leq c M_1^p (M_0 \sqrt{T})^{3p} & \mathbb{E}|D_r^i Y_{3,t}^\eta|^p &\leq c M_1^p (M_0 \sqrt{T})^{3p}
\end{aligned} \tag{B.7}$$

uniformly in $(r, s, t, u) \in [0, T]$ and $\eta \in [0, 1]$.

In the following lemma, we state a key result of the integration-by-parts formula allowing to represent the error term (4.9) using only $h^{(1)}$ and providing some moments' control useful in the error analysis.

Lemma B.2. *Let Assumptions (ELL), (RHO) and (\mathbb{R}_3) be in force. Let Z belong to $\cap_{p \geq 1} \mathbb{D}^{2,p}$. Then, for any $\eta \in [0, 1]$, for $k = 1, 2$, there exists a random variable Z_k^η in any $L^p(p \geq 1)$ such that for any function $l \in C_0^\infty(\mathbb{R}, \mathbb{R})$, one has*

$$\mathbb{E}[l^{(k)}(\eta Y_T + (1 - \eta) Y_T^0) Z] = \mathbb{E}[l(\eta Y_T + (1 - \eta) Y_T^0) Z_k^\eta]. \tag{B.8}$$

Moreover, one has $\|Z_k^\eta\|_p \leq c \frac{\|Z\|_{k,2p}}{((1-\rho^2)\lambda_{\inf}\sqrt{T})^k}$, uniformly in η .

Proof. We prove the lemma for $k = 1$, since for $k = 2$ the proof is similar.

Step 1: $F_\eta = \eta Y_T + (1 - \eta)Y_T^0$ is a non-degenerate random variable (in the Malliavin sense).

Under (\mathbb{R}_3) , we know that F_η is in $\cap_{p \geq 1} \mathbb{D}^{3,p}$. One has to prove that the Malliavin covariance matrix associated to F_η , which is a scalar in this case, is defined as

$$\gamma_{F_\eta} = \int_0^T (D_r^1 F_\eta)^2 dr + \int_0^T (D_r^2 F_\eta)^2 dr \quad (\text{B.9})$$

is almost surely positive and its inverse is in any $L^p(p \geq 1)$.

By linearity, we have

$$D_r^1 F_\eta = \eta D_r^1 Y_T + (1 - \eta) D_r^1 Y_T^0. \quad (\text{B.10})$$

From (B.1) and by setting η to 1 and 0 successively, we get for $r \leq t$

$$\begin{cases} D_r^1 Y_t &= \lambda(r, Y_r) \sqrt{1 - \rho^2} + \int_r^t D_r^1 Y_u (\alpha_y du + \lambda_y (\sqrt{1 - \rho^2} d\widetilde{W}_u^L + \rho dW_u^X)), \\ D_r^1 Y_t^0 &= \lambda(r, y_0) \sqrt{1 - \rho^2}. \end{cases} \quad (\text{B.11})$$

By solving (B.11) for $r \leq T$, we obtain

$$\begin{cases} D_r^1 Y_T &= \lambda(r, Y_r) \sqrt{1 - \rho^2} e^{\int_r^T \alpha_y du - \frac{1}{2} \int_r^T \lambda_y^2 du + \int_r^T \lambda_y (\sqrt{1 - \rho^2} d\widetilde{W}_u^L + \rho dW_u^X)} \\ D_r^1 Y_T^0 &= \lambda(r, y_0) \sqrt{1 - \rho^2}. \end{cases} \quad (\text{B.12})$$

Hence, we can write

$$\begin{aligned} \gamma_{F_\eta} &\geq \int_0^T (D_r^1 F_\eta)^2 dr \\ &\geq \int_0^T \left[\lambda(r, Y_r) \sqrt{1 - \rho^2} e^{\int_r^T \alpha_y du - \frac{1}{2} \int_r^T \lambda_y^2 du + \int_r^T \lambda_y (\sqrt{1 - \rho^2} d\widetilde{W}_u^L + \rho dW_u^X)} + (1 - \eta) \lambda(r, y_0) \sqrt{1 - \rho^2} \right]^2 dr \\ &\geq (\inf_{0 \leq r \leq T} e^{\int_r^T \alpha_y du - \frac{1}{2} \int_r^T \lambda_y^2 du + \int_r^T \lambda_y (\sqrt{1 - \rho^2} d\widetilde{W}_u^L + \rho dW_u^X)})^2 (1 - \rho^2) \int_0^T (\lambda(r, Y_r) + (1 - \eta) \lambda(r, y_0))^2 dr \\ &\geq (\inf_{0 \leq r \leq T} e^{\int_r^T \alpha_y du - \frac{1}{2} \int_r^T \lambda_y^2 du + \int_r^T \lambda_y (\sqrt{1 - \rho^2} d\widetilde{W}_u^L + \rho dW_u^X)})^2 T \lambda_{\inf}^2 (1 - \rho^2). \end{aligned} \quad (\text{B.13})$$

The second inequality shows that γ_{F_η} is almost surely positive. With the last inequality and the control of the moments for the solution of an SDE (see Priouret (2005, Section 6.2.1)), we get for $p \geq 1$

$$\|\gamma_{F_\eta}^{-1}\|_p \leq c \frac{1}{(\lambda_{\inf} \sqrt{1 - \rho^2} \sqrt{T})^2}. \quad (\text{B.14})$$

Step 2: Integration-by-parts formula.

Using Propositions 2.1.4 and 1.5.6 in Nualart (2005), one gets the existence of Z_1^η in $L^p(p \geq 1)$ with

$$\|Z_1^\eta\|_p \leq c \|\gamma_{F_\eta}^{-1}\|_{1,4p} \|DF_\eta\|_{1,4p} \|Z\|_{1,2p}. \quad (\text{B.15})$$

Step 3: Upper bound for $\|DF_\eta\|_{1,q}$, $\|\gamma_{F_\eta}^{-1}\|_{1,q}$, for $q \geq 2$.

We recall that

$$\|DF_\eta\|_{1,q}^q = \mathbb{E} \left[\left(\sum_{i=1}^2 \int_0^T (D_t^i F_\eta)^2 dt \right)^{\frac{q}{2}} \right] + \mathbb{E} \left[\left(\sum_{i,j=1}^2 \int_0^T (D_{t_i, t_j}^{i,j} F_\eta)^2 dt_i dt_j \right)^{\frac{q}{2}} \right]. \quad (\text{B.16})$$

We bound each term above separately using the linearity of the Malliavin derivative operator to F_η , the Holder inequality and the Malliavin derivatives' estimates in Lemma B.1, to obtain

$$\|DF_\eta\|_{1,q} \leq c \sqrt{T} |\lambda|_\infty. \quad (\text{B.17})$$

For $\|\gamma_{F_\eta}^{-1}\|_{1,q}$ which is given by

$$\|\gamma_{F_\eta}^{-1}\|_{1,q} = \mathbb{E} |\gamma_{F_\eta}^{-1}|^q + \mathbb{E} \left[\left(\sum_{i=1}^2 \int_0^T (D_t^i \gamma_{F_\eta}^{-1})^2 dt \right)^{\frac{q}{2}} \right], \quad (\text{B.18})$$

we use Lemma 2.1.6 in [Nualart \(2005\)](#) to write $D_t^i \gamma_{F_\eta}^{-1} = -\frac{D_t^i \gamma_{F_\eta}}{\gamma_{F_\eta}^2}$, $i = 1, 2$.

Similarly, we bound each term above separately using the linearity of the Malliavin derivative operator to γ_{F_η} (see [\(B.9\)](#)), the Hölder inequality, the Malliavin derivatives' estimates in Lemma [B.1](#) and the moments estimates in [\(B.14\)](#) to obtain

$$\|\gamma_{F_\eta}^{-1}\|_{1,q} \leq c \frac{1}{(\lambda_{\inf} \sqrt{1 - \rho^2} \sqrt{T})^2}. \quad (\text{B.19})$$

Finally, using $|\lambda|_\infty \leq C_E \lambda_{\inf}$ (Assumption [\(ELL\)](#)) combined with inequalities [\(B.17\)](#) and [\(B.19\)](#), we get

$$\|\gamma_{F_\eta}^{-1}\|_{1,4p} \|DF_\eta\|_{1,4p} \leq c \frac{1}{(1 - \rho^2) \lambda_{\inf} \sqrt{T}}. \quad (\text{B.20})$$

This completes our proof. \square

APPENDIX C. DERIVATION OF THE THIRD ORDER APPROXIMATION FORMULA

The additional correction terms for the third order expansion formula in [\(4.19\)](#) are provided by $\mathbb{E}[h^{(1)}(Y_T^0) \frac{Y_{2,T}}{2}]$ and $\mathbb{E}[h^{(2)}(Y_T^0) \frac{(Y_{1,T})^2}{2}]$. The first term, setting $\eta = 0$ in [\(3.18\)](#), yields

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \frac{Y_{2,T}}{2} \right] &= \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^T Y_{1,t} \alpha_y dt \right] + \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^T Y_{1,t} \lambda_y dW_t^L \right] \\ &+ \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^T Z_{1,t} \alpha_z dt \right] + \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^T \xi_t \gamma_t \alpha_{yz} dt \right] \\ &+ \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^T \frac{\xi_t^2}{2} \alpha_{yy} dt \right] + \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^T \frac{\gamma_t^2}{2} \alpha_{zz} dt \right] \\ &+ \mathbb{E} \left[h^{(1)}(Y_T^0) \int_0^T \frac{\xi_t^2}{2} \lambda_{yy} dW_t^L \right]. \end{aligned} \quad (\text{C.1})$$

Using Lemma [4.4](#), we compute each of the sub-correction terms separately:

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \left(\int_0^T Y_{1,t} \alpha_y dt \right) \right] &= [w(\alpha, \alpha_y, \alpha_y)_0^T + w(\beta, \alpha_z, \alpha_y)_0^T] g_1^h(Y_T^0) \\ &+ [w(\lambda^2, \alpha_y, \alpha_y)_0^T + w(\alpha, \lambda_y \lambda, \alpha_y)_0^T] g_2^h(Y_T^0) + w(\lambda^2, \lambda_y \lambda, \alpha_y) g_3^h(Y_T^0) \\ &+ \rho w(\sigma \lambda, \alpha_z, \alpha_y)_0^T g_2^h(Y_T^0), \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \left(\int_0^T Y_{1,t} \lambda_y dW_t^L \right) \right] &= [w(\alpha, \alpha_y, \lambda_y \lambda)_0^T + w(\beta, \alpha_z, \lambda_y \lambda)_0^T] g_2^h(Y_T^0) \\ &+ [w(\lambda^2, \alpha_y, \lambda_y \lambda)_0^T + w(\alpha, \lambda_y \lambda, \lambda_y \lambda)_0^T] g_3^h(Y_T^0) + w(\lambda^2, \lambda_y \lambda, \lambda_y \lambda)_0^T g_4^h(Y_T^0) \\ &+ \rho w(\sigma \lambda, \alpha_z, \lambda_y \lambda)_0^T g_3^h(Y_T^0), \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \left(\int_0^T Z_{1,t} \alpha_z dt \right) \right] &= w(\beta, \beta_z, \alpha_z)_0^T g_1^h(Y_T^0) \\ &+ \rho g_2^h(Y_T^0) [w(\sigma \lambda, \beta_z, \alpha_z)_0^T + w(\beta, \sigma_z \lambda, \alpha_z)_0^T] + \rho^2 w(\sigma \lambda, \sigma_z \lambda, \alpha_z)_0^T g_3^h(Y_T^0), \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \left(\int_0^T \xi_t \gamma_t \alpha_{yz} dt \right) \right] &= [\omega(\alpha, \beta, \alpha_{yz})_0^T + \omega(\beta, \alpha, \alpha_{yz})_0^T] g_1^h(Y_T^0) \\ &\quad + [\omega(\lambda^2, \beta, \alpha_{yz})_0^T + \omega(\beta, \lambda^2, \alpha_{yz})_0^T] g_2^h(Y_T^0) \\ &\quad + \rho [\omega(\lambda\sigma, \alpha_{yz})_0^T g_1^h(Y_T^0) + (\omega(\sigma\lambda, \alpha, \alpha_{yz})_0^T + \omega(\alpha, \sigma\lambda, \alpha_{yz})_0^T) g_2^h(Y_T^0) \\ &\quad + (\omega(\lambda^2, \sigma\lambda, \alpha_{yz})_0^T + \omega(\sigma\lambda, \lambda^2, \alpha_{yz})_0^T) g_3^h(Y_T^0)], \end{aligned} \quad (\text{C.5})$$

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \left(\int_0^T \frac{\xi_t^2}{2} \alpha_{yy} dt \right) \right] &= [w(\alpha, \alpha, \alpha_{yy})_0^T + \frac{1}{2} w(\lambda^2, \alpha_{yy})_0^T] g_1^h(Y_T^0) \\ &\quad + [w(\alpha, \lambda^2, \alpha_{yy})_0^T + w(\lambda^2, \alpha, \alpha_{yy})_0^T] g_2^h(Y_T^0) + w(\lambda^2, \lambda^2, \alpha_{yy})_0^T g_3^h(Y_T^0), \end{aligned} \quad (\text{C.6})$$

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \left(\int_0^T \frac{\gamma_t^2}{2} \alpha_{zz} dt \right) \right] &= [\frac{1}{2} w(\sigma^2, \alpha_{zz})_0^T + w(\beta, \beta, \alpha_{zz})_0^T] g_1^h(Y_T^0) \\ &\quad + \rho [w(\sigma\lambda, \beta, \alpha_{zz})_0^T + w(\beta, \sigma\lambda, \alpha_{zz})_0^T] g_2^h(Y_T^0) + \rho^2 w(\sigma\lambda, \sigma\lambda, \alpha_{zz})_0^T g_3^h(Y_T^0), \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} \mathbb{E} \left[h^{(1)}(Y_T^0) \left(\int_0^T \frac{\xi_t^2}{2} \lambda_{yy} dW_t^L \right) \right] &= \mathbb{E} \left[h^{(2)}(Y_T^0) \left(\int_0^T \frac{\xi_t^2}{2} \lambda_{yy} \lambda_t dt \right) \right] = \\ &= [\frac{1}{2} w(\lambda^2, \lambda\lambda_{yy})_0^T + w(\alpha, \alpha, \lambda\lambda_{yy})_0^T] g_2^h(Y_T^0) \\ &\quad + [w(\lambda^2, \alpha, \lambda\lambda_{yy})_0^T + w(\alpha, \lambda^2, \lambda\lambda_{yy})_0^T] g_3^h(Y_T^0) + w(\lambda^2, \lambda^2, \lambda\lambda_{yy})_0^T g_4^h(Y_T^0), \end{aligned} \quad (\text{C.8})$$

As for the second corrective term, by applying Itô's formula to $(Y_{1,t}^2)_{t \geq 0}$, we obtain

$$\begin{aligned} \mathbb{E} \left[\frac{h^{(2)}(Y_T^0)}{2} Y_{1,T}^2 \right] &= \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T Y_{1,t} \xi_t \alpha_y dt \right] + \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T Y_{1,t} \gamma_t \alpha_z dt \right] \\ &\quad + \mathbb{E} \left[\frac{h^{(2)}(Y_T^0)}{2} \int_0^T \xi_t^2 \lambda_y^2 dt \right] + \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T Y_{1,t} \xi_t \lambda_y dW_t^L \right] \\ &=: A + B + C + D. \end{aligned} \quad (\text{C.9})$$

We get directionally for C , by applying (4.42) with $\theta = \lambda_y^2$, that

$$\begin{aligned} C &= [\frac{1}{2} \omega(\lambda^2, \lambda_y^2)_0^T + \omega(\alpha, \alpha, \lambda_y^2)_0^T] g_2^h(Y_T^0) + [\omega(\lambda^2, \alpha, \lambda_y^2)_0^T + \omega(\alpha, \lambda^2, \lambda_y^2)_0^T] g_3^h(Y_T^0) \\ &\quad + \omega(\lambda^2, \lambda^2, \lambda_y^2)_0^T g_4^h(Y_T^0). \end{aligned} \quad (\text{C.10})$$

Moreover, by applying Lemma 4.3 to D , we get

$$D = \mathbb{E} \left[h^{(3)}(Y_T^0) \int_0^T Y_{1,t} \xi_t \lambda_y \lambda dt \right], \quad (\text{C.11})$$

hence A and D can be computed similarly. Indeed, with Itô's formula and successive applications of Lemmata 4.2 and 4.3, we obtain for D

$$\begin{aligned}
D &= \mathbb{E} \left[h^{(3)}(Y_T^0) \int_0^T Y_{1,t} \alpha \omega (\lambda \lambda_y)_t^T dt \right] + \mathbb{E} \left[h^{(3)}(Y_T^0) \int_0^T \xi_t^2 \alpha_y \omega (\lambda \lambda_y)_t^T dt \right] \\
&+ \mathbb{E} \left[h^{(3)}(Y_T^0) \int_0^T \xi_t \gamma_t \alpha_z \omega (\lambda \lambda_y)_t^T dt \right] + \mathbb{E} \left[h^{(3)}(Y_T^0) \int_0^T \xi_t \lambda \lambda_y \omega (\lambda \lambda_y)_t^T dt \right] \\
&+ \mathbb{E} \left[h^{(4)}(Y_T^0) \int_0^T Y_{1,t} \lambda^2 \omega (\lambda \lambda_y)_t^T dt \right] + \mathbb{E} \left[h^{(4)}(Y_T^0) \int_0^T \xi_t^2 \lambda \lambda_y \omega (\lambda \lambda_y)_t^T dt \right] \\
&=: D_1 + D_2 + D_3 + D_4 + D_5 + D_6.
\end{aligned} \tag{C.12}$$

Each term D_i is computed explicitly using Lemma 4.4. Furthermore, with

$$\alpha_y(t, y, z) = -[\lambda_y(t, y) \lambda(t, y) + \rho \lambda_y(t, y) \sigma(t, z)] \tag{C.13}$$

we deduce directly the following expression for A :

$$\begin{aligned}
A &= \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T Y_{1,t} \xi_t \alpha_y dt \right] \\
&= -\mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T Y_{1,t} \xi_t \lambda_y \lambda dt \right] - \rho \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T Y_{1,t} \xi_t \lambda_y \sigma dt \right].
\end{aligned} \tag{C.14}$$

Each term above can be deduced from D . Finally, the expression for B is given by

$$\begin{aligned}
B &= \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T Y_{1,t} \beta \omega (\alpha_z)_t^T dt \right] + \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T \gamma_t^2 \alpha_z \omega (\alpha_z)_t^T dt \right] \\
&+ \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T \xi_t \gamma_t \alpha_y \omega (\alpha_z)_t^T dt \right] + \rho \mathbb{E} \left[h^{(2)}(Y_T^0) \int_0^T \xi_t \sigma \lambda_y \omega (\alpha_z)_t^T dt \right] \\
&+ \rho \mathbb{E} \left[h^{(3)}(Y_T^0) \int_0^T Y_{1,t} \lambda \sigma \omega (\alpha_z)_t^T dt \right] + \mathbb{E} \left[h^{(3)}(Y_T^0) \int_0^T \xi_t \gamma_t \lambda \lambda_y \omega (\alpha_z)_t^T dt \right] \\
&=: B_1 + B_2 + B_3 + B_4 + B_5 + B_6.
\end{aligned} \tag{C.15}$$

By gathering all these terms, passing to the initial parameters and writing them as a polynomial function of ρ (order 4), we obtain the third order expansion formulas in (3.29). We omit the details of these computations for the sake of brevity and provide directly the results. The constant coefficient is as in Theorem 3.7. The other coefficients are provided below.

We have that

$$\gamma_{1,T} = \sum_{j=1}^5 \gamma_{1,j,T} g_j^h(Y_T^0) \quad (\text{C.16})$$

$$\begin{aligned} \gamma_{1,1,T} &= A_{7,T} + \frac{1}{2}(A_{8,T} + A_{9,T} - A_{10,T} - A_{11,T}) - B_{28,T} \\ &\quad - \frac{1}{2}(B_{26,T} + B_{27,T} + B_{32,T} + B_{36,T} + B_{35,T} + B_{31,T} + B_{42,T}) \\ &\quad - \frac{1}{4}(B_{33,T} + B_{34,T} + B_{29,T} + B_{37,T}) \end{aligned} \quad (\text{C.17})$$

$$\begin{aligned} \gamma_{1,2,T} &= -A_{7,T} - A_{8,T} + B_{29,T} + \frac{5}{2}B_{27,T} + 3(B_{28,T} + B_{26,T}) + \frac{1}{2}(B_{32,T} + B_{33,T} + B_{34,T}) \\ &\quad + \frac{3}{2}(B_{31,T} + B_{30,T} + B_{42,T} + B_{35,T}) + C_{56,T} + C_{55,T} + \frac{1}{2}(C_{50,T} + C_{52,T} + C_{54,T} \\ &\quad + C_{78,T}) + \frac{1}{4}(C_{5,T} + C_{6,T} + C_{7,T} + C_{51,T} + C_{53,T} + C_{57,T} + C_{58,T} + C_{77,T}) \end{aligned} \quad (\text{C.18})$$

$$\begin{aligned} \gamma_{1,3,T} &= -3B_{26,T} - 2(B_{27,T} + B_{28,T}) - (B_{29,T} + B_{30,T} + B_{31,T} + B_{35,T} + B_{42,T}) \\ &\quad - \frac{3}{4}(C_{5,T} + C_{6,T} + C_{7,T} + C_{53,T} + C_{57,T} + C_{58,T}) - 4(C_{55,T} + C_{56,T}) \\ &\quad - 2(C_{50,T} + C_{52,T}) - \frac{5}{2}(C_{54,T} + C_{78,T}) - \frac{5}{4}(C_{51,T} + C_{77,T}) \end{aligned} \quad (\text{C.19})$$

$$\begin{aligned} \gamma_{1,4,T} &= \frac{1}{2}(C_{5,T} + C_{6,T} + C_{7,T} + C_{57,T} + C_{58,T} + C_{53,T}) + 2(C_{51,T} + C_{77,T}) \\ &\quad + 4(C_{54,T} + C_{78,T}) + 5(C_{55,T} + C_{56,T}) + \frac{5}{2}(C_{50,T} + C_{52,T}) \end{aligned} \quad (\text{C.20})$$

$$\gamma_{1,5,T} = -C_{50,T} - C_{51,T} - C_{52,T} - C_{77,T} - 2(C_{54,T} + C_{55,T} + C_{56,T} + C_{78,T}), \quad (\text{C.21})$$

and

$$\gamma_{2,T} = \sum_{j=1}^5 \gamma_{2,j,T} g_j^h(Y_T^0) \quad (\text{C.22})$$

$$\begin{aligned} \gamma_{2,1,T} &= A_{4,T} - A_{5,T} - \frac{1}{2}(B_{10,T} + B_{13,T} + B_{14,T} + B_{15,T} + B_{16,T} + B_{17,T}) \\ &\quad - B_{11,T} - B_{12,T} - B_{18,T} - B_{19,T} \end{aligned} \quad (\text{C.23})$$

$$\begin{aligned} \gamma_{2,2,T} &= -A_{6,T} + 2(B_{10,T} + B_{20,T} + B_{11,T}) + B_{12,T} + B_{41,T} + B_{16,T} + B_{17,T} + B_{18,T} \\ &\quad + B_{19,T} + B_{38,T} + \frac{3}{2}B_{21,T} + \frac{1}{2}(B_{22,T} + B_{23,T} + B_{24,T} + B_{25,T} + B_{39,T}) \\ &\quad + C_{39,T} + C_{42,T} + C_{43,T} + C_{84,T} + C_{85,T} \\ &\quad + 2C_{44,T} + \frac{1}{4}(C_{8,T} + C_{10,T} + C_{12,T} + C_{14,T} + C_{80,T} + C_{82,T} + C_{86,T} + C_{87,T}) \\ &\quad + \frac{1}{2}(C_{9,T} + C_{11,T} + C_{15,T} + C_{16,T} + C_{38,T} + C_{40,T} + C_{41,T} + C_{45,T} + C_{46,T} \\ &\quad + C_{79,T} + C_{81,T} + C_{83,T}) \end{aligned} \quad (\text{C.24})$$

$$\begin{aligned}
\gamma_{2,3,T} = & -(B_{23,T} + B_{22,T} + B_{38,T} + B_{39,T}) - 2B_{20,T} \\
& - C_{80,T} - 2(C_{83,T} + C_{39,T}) - 3(C_{42,T} + C_{43,T} + C_{84,T} + C_{85,T}) - 4C_{44,T} \\
& - \frac{1}{2}(C_{8,T} + C_{9,T} + C_{17,T} + C_{15,T} + C_{16,T} + C_{12,T} + C_{14,T} + C_{18,T} + C_{19,T}) \\
& - \frac{3}{2}(C_{38,T} + C_{40,T} + C_{79,T} + C_{81,T}) \\
& - \frac{1}{2}(C_{41,T} + C_{45,T} + C_{46,T} + C_{47,T} + C_{48,T} + C_{49,T} + C_{86,T} + C_{87,T} + C_{82,T})
\end{aligned} \tag{C.25}$$

$$\begin{aligned}
\gamma_{2,4,T} = & C_{38,T} + C_{39,T} + C_{40,T} + C_{79,T} + C_{80,T} + C_{81,T} \\
& + 2(C_{44,T} + C_{42,T} + C_{43,T} + C_{85,T} + C_{83,T} + C_{84,T}) \\
& + \frac{3}{2}(C_{17,T} + C_{19,T} + C_{18,T} + C_{47,T} + C_{48,T} + C_{49,T})
\end{aligned} \tag{C.26}$$

$$\gamma_{2,5,T} = -(C_{17,T} + C_{19,T} + C_{18,T} + C_{49,T} + C_{48,T} + C_{47,T}), \tag{C.27}$$

and

$$\gamma_{3,T} = \sum_{j=1}^4 \gamma_{3,j,T} g_j^h(Y_T^0) \tag{C.28}$$

$$\gamma_{3,1,T} = -B_{4,T} - B_{8,T} \tag{C.29}$$

$$\begin{aligned}
\gamma_{3,2,T} = & 2(B_{5,T} + B_{7,T} + B_{40,T}) + C_{64,T} + C_{67,T} + C_{68,T} + C_{34,T} + 2(C_{69,T} + C_{35,T}) \\
& + \frac{1}{2}(C_{20,T} + C_{21,T} + C_{22,T} + C_{63,T} + C_{65,T} + C_{66,T} + C_{70,T} + C_{71,T})
\end{aligned} \tag{C.30}$$

$$\begin{aligned}
\gamma_{3,3,T} = & -B_{6,T} - B_{9,T} - (C_{31,T} + C_{24,T} + C_{25,T} + C_{34,T} + C_{63,T} + C_{64,T} + C_{65,T} + C_{36,T}) \\
& - 2(C_{27,T} + C_{35,T} + C_{69,T} + C_{67,T} + C_{68,T} + C_{37,T}) \\
& - \frac{1}{2}(C_{26,T} + C_{23,T} + C_{28,T} + C_{29,T} + C_{30,T} + C_{73,T} + C_{74,T} + C_{75,T})
\end{aligned} \tag{C.31}$$

$$\begin{aligned}
\gamma_{3,4,T} = & C_{28,T} + C_{26,T} + C_{30,T} + C_{31,T} + C_{75,T} + C_{74,T} + C_{73,T} + C_{36,T} \\
& + 2(C_{27,T} + C_{37,T}),
\end{aligned} \tag{C.32}$$

and finally

$$\gamma_{4,T} = \sum_{j=2}^4 \gamma_{4,j,T} g_j^h(Y_T^0) \tag{C.33}$$

$$\gamma_{4,2,T} = C_{59,T} + 2C_{60,T} \tag{C.34}$$

$$\gamma_{4,3,T} = -C_{4,T} - C_{61,T} - 2(C_{62,T} + C_{3,T}) \tag{C.35}$$

$$\gamma_{4,4,T} = 2C_{1,T} + C_{2,T} \tag{C.36}$$

All the expressions for the coefficients $A_{i,T}$, $B_{i,T}$ and $C_{i,T}$ are gathered in Tables C.1, C.2 and C.3 below.

$A_{1,T} = \omega(\lambda^2, \lambda\lambda_y)_0^T$	$A_{2,T} = \omega(\lambda^2, \lambda_{yy}\lambda)_0^T$	$A_{3,T} = \omega(\lambda^2, \lambda_y^2)_0^T$	$A_{4,T} = \omega(\lambda\sigma, \lambda_y\sigma)_0^T$
$A_{5,T} = \omega(\lambda\sigma, \lambda_y\sigma_z)_0^T$	$A_{6,T} = \omega(\lambda\sigma, \lambda\sigma_z)_0^T$	$A_{7,T} = \omega(\lambda\sigma, \lambda_y\lambda)_0^T$	$A_{8,T} = \omega(\lambda^2, \lambda_y\sigma)_0^T$
$A_{9,T} = \omega(\sigma^2, \lambda\sigma_z)_0^T$	$A_{10,T} = \omega(\lambda^2, \lambda_{yy}\sigma)_0^T$	$A_{11,T} = \omega(\sigma^2, \lambda\sigma_{zz})_0^T$	

TABLE C.1. Weight coefficients involving 2 multiple integrals

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