UTILITY INDIFFERENCE PRICING OF DERIVATIVES WRITTEN ON INDUSTRIAL LOSS INDEXES

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ABSTRACT. We consider the problem of pricing derivatives written on some industrial loss index via utility indifference pricing. The industrial loss index is modeled by a compound Poisson process and the insurer can adjust her portfolio by choosing the risk loading, which in turn determines the demand. We compute the price of a CAT (spread) option written on that index using utility indifference pricing.

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1. Introduction

It was recognized shortly after the Hurricane Andrew in 1992, then the most costly natural catastrophe in history, that events of this magnitude significantly stress the capacity of the insurance industry. On the other hand, the accumulated losses of those events are rather small relative to the US stock and bond markets. Thus, securitization offers a potentially more efficient mechanism for financing CAT losses than conventional insurance and reinsurance, see Cummins et al, [CLP04].

The first contracts were launched by the Chicago Board of Trade (CBOT), which introduced catastrophe futures in 1992 and later introduced catastrophe put and call options. The options were based on aggregate catastrophe-loss indices compiled by Property Claims Services, an insurance industry statistical agent, see [Cum06].

In the absence of a traded underlying asset, insurance-linked securities have been structured to pay-off on three types of variables: Insurance-industry catastrophe loss indices, insurer-specific catastrophe losses, and parametric indices based on the physical characteristics of catastrophic events. The first variant involves higher basis risk and less exposure to moral hazard than the second, the third variant tries to balance the two risks in a suitable way, cf. Cummins

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[Cum06]. In this paper we solely concentrate on index-based derivatives.

A simple example of such a derivative is provided by the aforementioned call options on an insurance-industry catastrophe loss index. The variant introduced by CBOT was actually a call option spread, that is, a combination of a call option long and another call option short with a higher strike.

A more popular type of catastrophe derivative is the CAT bond. This is a classical bond in which there is an option embedded which is triggered by a defined catastrophic event. In this paper we will again only consider those bonds where this catastrophic event depends on some industry-loss index, though in practice both of the other variants are of importance as well. From our point of view there is little difference between CAT bond and CAT option, since on evaluating a CAT bond we concentrate on the embedded option. There is however some danger of confusion regarding the role of buyer/seller with CAT bonds: The issuer of the bond actually buys the embedded option while the buyer of the bond sells the option.

For the issuer of a CAT bond – typically an insurance or reinsurance company – it serves as a reinsurance. On the other hand, the investor who buys the bond (and therefore sells an option) receives a coupon over the market interest and can, at the same time, diversify her risk by investing in a security whose payoff is largely uncorrelated with classical financial instruments.

Geman and Yor [GY97] analyze catastrophe options with payoff $(C(T)-K)^+$ where C is the aggregate claims process which is modeled by a jump-diffusion process. Cox [CFP04] used a pure Poisson process to model the aggregate loss of an insurance company, and derived the pricing formula of CATEputs under the assumptions of constant arrival rates of catastrophic events. Jaimungal and Wang [JW06] used a compound Poisson process to describe the dynamic losses more accurately, but maintain the assumption of the constant arrival rate of claims.

We model the arrival of claims, which are accounted for in some industrial loss index, as a Poisson process with fixed arrival intensity. The underlying of the CAT derivative, the index, is itself not tradable. It therefore makes sense to use the method of indifference pricing via expected utility of Hodges and Neuberger [HNb89] to price the derivative. A similar approach can be found in Egami and Young [EY08], where the authors used utility indifference pricing techniques to price structured catastrophe bonds. However, there is a big difference in our modeling of the hedging opportunity. In our setup this is done via adjusting the insured portfolio.

For catastrophic events, the assumption that the resulting claims occur at jump times of a Poisson process as adopted by most previous studies is not beyond justifiable critique. Therefore alternative point processes have been used to generate the claim arrival process. Lin et al [LCP09] proposed a doubly stochastic Poisson process, (also called "Cox process", see [CFP04, Gra76, Gra91, Bre81, Lan94]) to model the arrival process for catastrophic events and derived pricing formulas of contingent capital. See also Fuita et al [FIT08] for arbitrage pricing of CAT bonds in such a context. Jaimungal and Chong [JC14] consider valuation of catastrophe derivatives when the rate of the claims is modulated by a Markov chain.

Charpentier [Cha08] considers hedging of catastrophe derivatives with stocks whose jumps depend on catastrophic events and how to compute a utility indifference price in this setup.

Our study contributes to the literature by presenting a new approach to hedging a CAT derivative via adjustment of the insured portfolio, which in turn is done via adjusting the risk loading and an exogenously given demand curve. The main idea is that the loss in the portfolio of a single insurance company is necessarily correlated with an industrial loss index that includes the losses of that insurance. The introduction of the derivative has therefore an influence on the pricing policy of the insurance company.

It has been noted by Cummins [Cum06] that the relatively low volume in the CAT derivatives market may in part be due to insufficient understanding of how these products may be hedged. Our paper gives a new perspective to the hedging of CAT derivatives via the most basic operation of an insurance company, i.e. the choice of a suitable risk loading for a particular risk. Future work may combine this approach with other hedging methods, like trading in shares that are correlated with catastrophic events, such as those of construction companies.

The paper is organized as follows: In Section 2 we give the problem description: We assume a global claims process C, which keeps track of all claims due to a specific event in a given country and we consider an insurance company in the same country, so that the index will contain the losses of that particular insurance company among others. The insurance company is facing a certain demand curve which determines the fraction of the insurance market that the company gets to insure, dependent on the risk loading it charges. We therefore have to model the ξ-fraction of the claims process C, for an insurance company with a ξ-fraction of the market. Such a model is constructed in Section 2 where we also derive the wealth process for the insurance company. We conclude that section by giving a short introduction into the concept of utility indifference pricing.

Section 3 constitutes the main part of our paper: We derive a suitable Hamilton-Jacobi-Bellman (HJB) equation for an insurance that holds k units of a derivative written on the total number of claims (Subsection 3.1). We use the concept of *piecewise deterministic Markov decision process* as presented, e.g., by Bäuerle and Rieder in [BR10, BR11]. In particular, we will make use of a verification theorem from [BR11] to show that a solution to the HJB equation also solves the optimal control problem. At that stage we will have to specialize to exponential utility.

Two subsections, 3.3 and 3.4, are devoted to special demand functions. While Subsection 3.3 looks into the details of the very special case of linear demand, Subsection 3.4 presents a class of demands that are more general than the previous linear one, but still preserve the property of leading to a unique optimal risk loading. By the end of Section 2.4, in Subsection 3.5, we give a numerical example for linear demand. In the technical Subsection 3.6 we show that the conditions of the verification theorem are satisfied.

Section 4 is devoted to the question under which conditions the derivative could actually be sold, that is, when the buyer's price is at least as big as the seller's price. To that end we study a couple of different pricing concepts related to the utility indifference price.

2. Model setup

2.1. **Study problem.** Suppose we have a global claims process $C = (C_t)_{t\geq 0}$, which keeps track of all insurance claims due to a specific type of event in a country. That is, C_t is the cumulative sum of all claims up to time t. We assume that there are M possible clients in the market which potentially contribute to the claims process. If all those clients had insurance contracts with the same insurance company, then C were the claims process of this insurance company. Let a be the "fair" annual premium for one client, that is $\mathbb{E}(C_1) = M \cdot a$. The annual premium for one contract therefore has to be greater or equal than a, since otherwise the insurance will make an almost sure loss in the long run.

Assume that an insurance company faces a demand curve q such that if the premium the insurance charges for the claim is $a(1+\theta)$, then the company gets to insure the $\frac{q(\theta)}{M}$-th part of the whole claim process for the total annual premium $a(1+\theta)q(\theta)$, where M is the total number of clients. It is assumed that q is continuous and strictly decreasing in θ. We further make the reasonable assumption that the insurance gets to insure the whole process if it does not charge any risk loading (any strictly risk-averse client would enter such a contract) and it gets 0 contracts if the risk-loading exceeds some fixed number $m > 0$. In our model θ may vary over time and we assume that the number of contracts is adjusted instantaneously via the demand function q. This is a simplifying assumption that will not be met in practice. However we will see in our numerical examples that there are large areas where the risk loading is almost constant over time and the index value. Thus our analysis can either be viewed as a first order approximation to more realistic models or as a model that has practical relevance only under certain market conditions.

Note that in the above setting the insurance company is not necessarily a monopolist: q is the demand that the company faces, which may well be influenced by other insurance companies' decisions. We only assume that competing firms respond consistently to changes in the risk loading, so that q does not change over time and is not influenced by earlier policy decisions. We will ignore the influence of the derivative that we want to price on demand, i.e. we assume that the demand the insurance company faces on its insurance contracts is the same regardless of whether the insurance company introduces the derivative or not.

One question arising here is how we can model the ξ-th part of the industry loss process in a way that an insurance company which holds contracts for the

ξ-th part of the market will only be confronted with the ξ-th part of the claims? For fixed ξ this will be a thinning of the original process. Once we have found a model for this, we find that the wealth process of the insurance company can be controlled via the risk loading process and therefore we can ask for optimal strategies for maximizing terminal utility. This in turn will make it possible to use the method of utility indifference pricing for CAT-derivatives.

Since the wealth process is obviously correlated to the global claims process C, any derivative written on C_T, for some fixed $T > 0$, can be partially hedged. This will result in a utility indifference price that is different from the utility equivalence price.

2.2. **Exposure to industry loss.** We assume as given a Poisson process N with intensity λM, which models the arrivals of claims, as well as sequences of i.i.d. random variables $Y_1, Y_2, \ldots$ with values in $\mathbb{R}_+$, the sizes of the claims. Moreover we assume that there are i.i.d. random variables $U_1, U_2, \ldots$ with values in the space of all possible insurance clients. The Y_k's and the U_k's are assumed to be independent of each other and independent of N. That is, N tells us at which time τ_k the k-th claim occurs, Y_k models its size and U_k tells us who is affected. Actually, from the point of view of an insurance company the only interesting information about U_k is whether it is one of their own clients who is affected or not. We will therefore assume that the U_k's are uniformly distributed on $[0, 1]$ and that a particular insurance company which holds the ξ-th part of the market is therefore affected with probability ξ.

Hence the claims process for this insurance company can be modeled by

$$C_t^\xi := \sum_{k=1}^{N_t} Y_k 1_{U_k \le \xi_{\tau_k}} .$$

The cumulative claims process of all claims constitutes our industrial loss index and is defined as

$$C_t := C_t^1 = \sum_{k=1}^{N_t} Y_k .$$

Note that, for constant $\xi \in [0, 1]$, the process C^ξ is a *thinning* of C and therefore is a compound Poisson process with intensity $\xi\lambda M$ (see, e.g., [Res92, Section 4.4]). The joint process (C, C^ξ) is therefore a compound Poisson process with values in $[0, \infty) \times [0, \infty)$ and with jump distribution

$$\mathbb{P}((\Delta C_{\tau_k}, \Delta C_{\tau_k}^\xi) \in A) = \xi \mathbb{P}((Y_k, Y_k) \in A) + (1 - \xi)\mathbb{P}((Y_k, 0) \in A) .$$

for all Borel measurable sets $A \in [0, \infty) \times [0, \infty)$.

2.3. **Demand function and wealth process.** We now turn to the function q which determines the fraction of the market that the company gets to insure depending on the risk loading it chooses. We assume that q is a strictly decreasing continuous function on $\mathbb{R}$ with $q(\theta) = M$ for $\theta \le 0$ and $q(\theta) = 0$ for $\theta \ge m$, where m is some positive real number. Therefore q is rather general, even the requirement that it vanishes above some level m is rather innocuous: suppose the annual premium were much larger than the expected claim size, then surely this insurance could not be sold.

We further define $a := \lambda\mathbb{E}(Y_1)$, the expected annual claim per client. In particular we assume $\mathbb{E}(Y_1) < \infty$. In fact we will have to assume the stronger assumption $\mathbb{E}(e^{bY_1}) < \infty$ for some given constant $b > 0$.

For a measurable function $\theta : [0, \infty) \to \mathbb{R}$ we therefore define the dynamics of the wealth process for the insurance company as

$$X_t^\theta := x_0 + \int_0^t a(1+\theta_s)q(\theta_s)ds - \sum_{k=1}^{N_t} Y_k 1_{U_k \le q(\theta_{\tau_k})/M}\,, \tag{1}$$

$$= x_0 + \int_0^t a(1+\theta_s)q(\theta_s)ds - C_t^{q(\theta)/M} \tag{2}$$

where q is the absolute demand function as introduced above, and x_0 is the initial wealth.

The process (C, X^θ) is a special case of a *piecewise deterministic Markov process*, where the flow does not depend on X. See [BR11, Chapter 8] or [BR10] for the definition and theory of piecewise deterministic Markov processes. The corresponding data is given by

- the state space $[0, \infty) \times \mathbb{R}$;
- the control space $\mathbb{R}$ (we will later see that we may restrict the controls to the compact space $[0, m]$);
- the deterministic flow $d(C_t, X_t^\theta) = (0, aq(\theta_t)(1+\theta_t))dt$ between jumps;
- the jump intensity $M\lambda$;
- the stochastic kernel Q,

$$Q(A|(c,x),\theta) = q(\theta)\mathbb{P}((Y,-Y) \in A - (c,x)) + (1-q(\theta))\mathbb{P}((Y,0) \in A - (c,x))\,,$$

  where $A - (c,x)) = \{(a_1 - c, a_2 - x) : (a_1, a_2 \in A)\}$;
- the zero reward rate;
- the discount rate, which we set 0 for simplicity.

Here and throughout the paper, Y denotes a random variable having the same distribution as the Y_k.

A Markovian control for this system is a measurable function

$$f : [0,\infty) \times \mathbb{R} \times [0,\infty) \longrightarrow \{\theta : [0,\infty) \to \mathbb{R}, \theta \text{ measurable}\}$$

which describes for input data (C_τ, X_τ, τ) — where τ is the time of a jump and (C_τ, X_τ) is the state of the process immediately after the jump — the control until the next jump.

2.4. **Utility and utility indifference pricing.** Throughout the paper we assume that investors and insurance companies have a utility function, by which we mean a function $u : \mathbb{R} \longrightarrow \mathbb{R}$ which is strictly increasing and concave, and that they aim to maximize the expected utility of their wealth at some time T in the future.

We recall the general idea of utility indifference pricing as introduced in Hodges and Neuberger [HNb89]. An excellent introduction is Henderson and Hobson [HH09], from which we will repeat the basic definitions.

The utility indifference buy (or bid) price p^b is the price at which the investor is indifferent (in the sense that his expected utility under optimal trading is unchanged) between paying nothing and not having the claim Z_T and paying p^b now and receiving the claim Z_T at time T.

Consider the problem with $k \geq 0$ units of the claim. Assume an investor with utility u who initially has wealth x and zero endowment. Define

$$V(x,k) := \sup_{X_T \in \mathcal{A}(x)} \mathbb{E}(u(X_T + kZ_T))$$

where $\mathcal{A}(x)$ is the set of all wealths X_T which can be generated from initial fortune x by following admissible strategies. The *utility indifference buy price* $p^b(k)$ is the solution to

$$V(x - p^b(k), k) = V(x, 0)\,. \tag{3}$$

That is, the investor is willing to pay at most the amount $p^b(k)$ today for k units of the claim Z_T at time T. Similarly the *utility indifference sell price* $p^s(k)$ is the smallest amount the investor is willing to accept in order to sell $k \geq 0$ units of Z_T. That is, $p^s(k)$ solves

$$V(x + p^s(k), -k) = V(x, 0)\,.$$

The two prices are related via $p^b(k) = -p^s(-k)$. With this in mind we can define the *utility indifference price* $p(k)$ as the solution to (3) for all $k \in \mathbb{R}$.

Henderson and Hobson [HH09] note two features of the utility indifference price.

- *Non-linear pricing:* In contrast to the Black-Scholes price (and many alternative pricing methodologies in incomplete markets), utility prices are generally non-linear in the number of claims, i.e. k.
- *Recovery of complete market price:* If the market is complete or if the claim Z_T is replicable, the utility indifference price $p(k)$ is equal to the complete market price for k units of the claim.

It should be noted that, in general, the utility indifference price $p(k)$ also depends on x. This dependence usually vanishes for exponential utility: Suppose that $u(x) = -\exp(-\eta x)$, for some $\eta > 0$. Then

$$\begin{aligned} V(x - p(k), k) &= \sup_{\theta \in \Theta} \mathbb{E}\left(u(x - p(k) + G_T^\theta + kZ_T)\right) \\ &= -e^{-\eta x} e^{\eta p(k)} \inf_{\theta \in \Theta} \mathbb{E}\left(\exp(-\eta(G_T^\theta + kZ_T))\right)\,, \\ V(x, 0) &= -e^{-\eta x} \inf_{\theta \in \Theta} \mathbb{E}\left(\exp(-\eta(G_T^\theta))\right)\,, \end{aligned}$$

such that, with our former notation

$$p(k) = -\frac{1}{\eta}\left(\log\left(\inf_{\theta \in \Theta} \mathbb{E}(\exp(-\eta(G_T^\theta + kZ_T)))\right) - \log\left(\inf_{\theta \in \Theta} \mathbb{E}(\exp(-\eta G_T^\theta))\right)\right) \tag{4}$$

(provided that the arguments in the logarithms are finite).

See also [Bec03] for additional properties of the utility indifference price for exponential utility.

We mention another feature of the utility indifference price: Suppose the payment Z_T is independent of G_T^θ for every choice of θ. Then

$$\mathbb{E}(\exp(-\eta(G_T^\theta + kZ_T))) = \mathbb{E}(\exp(-\eta G_T^\theta)\mathbb{E}(\exp(-\eta kZ_T))$$

and therefore

$$p(k) = -\frac{1}{\eta}\log \mathbb{E}(\exp(-\eta kZ_T))\,.$$

This price is also called the *certainty equivalence price* of the derivative Z_T. Note that one special case where Z_T is independent of G_T^θ occurs when G_T^θ is deterministic.

3. Computation of the utility indifference price

3.1. **Optimal dynamic risk loading.** We now want to apply the concept of utility indifference pricing to the model presented in Section 2. Consider a derivative written on C_T, the total claims process at time T. Let its payoff be of the form $\psi(C_T)$ where ψ is a continuous and bounded function on $[0,\infty)$. For example, if the derivative is a CAT (spread) option then ψ has the form

$$\psi(c) = \max(0, \min(c-K, L-K))\,.$$

For a given utility function u we want to maximize expected utility from terminal wealth, i.e. we want to compute

$$\sup_\theta \mathbb{E}(u(X_T^\theta + k\psi(C_T)))\,,$$

where θ ranges over all Markovian controls.

We have the following simple lemma which allows us to limit our considerations to bounded θ:

Lemma 3.1. *Let $\theta : [0,\infty) \to \mathbb{R}$ be measurable. Define κ by*

$$\kappa = \begin{cases} 0 & \text{if} \quad \theta < 0 \\ \theta & \text{if} \quad 0 \le \theta \le m \\ m & \text{if} \quad \theta > m\,. \end{cases}$$

Then κ is measurable with values in $[0,m]$ and for all $\tau_n \le t < \tau_{n+1}$

$$X_t^\kappa - X_{\tau_n}^\kappa \ge X_t^\theta - X_{\tau_n}^\theta\,.$$

Proof. It holds that $q(\kappa) = q(\theta)$ throughout by our assumptions on q. We therefore have

$$\begin{aligned} X_t^\kappa - X_{\tau_n}^\kappa - X_t^\theta + X_{\tau_n}^\theta &= \int_{\tau_n}^t a(1+\kappa_s)q(\kappa_s) - a(1+\theta_s)q(\theta_s)ds \\ &= a\int_{\tau_n}^t (\kappa_s - \theta_s)q(\kappa_s) \ \ge 0\,, \end{aligned}$$

since $\kappa \ge \theta$ if $\theta \le m$ and $q(\kappa) = 0$ if $\theta > m$. □

Define the value function of the problem as

$$V(t,x,c,k) := \sup_{\theta} \mathbb{E}(u(X_T^\theta + k\psi(C_T))|X_t^\theta = x, C_t = c)\,, \tag{5}$$

where the supremum is taken over all Markovian controls θ. According to Lemma 3.1 we may concentrate on θ with values in $[0,m]$. Note that due to the boundedness of ψ we have for $\theta \equiv m$ that $\mathbb{E}(u(X_T^\theta + k\psi(C_T))|X_t^\theta = x, C_t = c) > -\infty$ for all x, c, k, such that $V(t,x,c,k) > -\infty$. Furthermore the expected terminal wealth X_T is bounded from above, since the growth rate of X is bounded and only negative jumps can occur. V is therefore a well-defined real-valued function.

[BR10] prove that under fairly general conditions there exists an optimal *relaxed* control for finite horizon problems for piecewise deterministic Markov decision processes. They also give conditions under which there exists a *non-relaxed* policy. Those later conditions are not satisfied for our problem, but relaxed controls are to weak for the purpose of utility indifference pricing. We therefore take a slightly different path using a Hamilton-Jacobi-Bellman (HJB) equation and a slight variation of the verification theorem [BR11, Theorem 8.2.8].

Definition 3.2. A measurable function $b : [0,\infty) \times \mathbb{R} \longrightarrow [0,\infty)$ is called a *bounding function* for the piecewise deterministic Markov decision model, if there exist constants $c_u, c_Q, c_{\text{flow}} \geq 0$ such that for all $c \in [0,\infty), x \in \mathbb{R}$

(i) $|u(c,x)| \leq c_u b(c,x)$;
(ii) $\int b(c,x) Q(dc \times dx|c,x,\theta) \leq c_Q b(c,x)$ for all $\theta \in [0,m]$;
(iii) $b(c, \int_0^T \int_0^m aq(y)(1+y)\alpha_s(dy)ds) \leq c_{\text{flow}} b(c,x)$ for all $\alpha \in \mathcal{R}$.

Here $\mathcal{R}$ is the space of *relaxed policies*, i.e. of measurable maps $[0,\infty) \to \mathcal{P}([0,m])$, where $\mathcal{P}([0,m])$ is the space of all probability measures on the Borel σ-algebra on $[0,m]$. See again [BR10].

Definition 3.3. Let $b : [0,\infty) \times \mathbb{R} \longrightarrow [0,\infty)$ be a measurable function. For some fixed $\gamma > 0$ we define

$$b(c,x,t) := b(c,x)\exp(\gamma(T-t))\,.$$

Further we define, for any measurable function $v : [0,\infty) \times \mathbb{R} \times [0,T] \longrightarrow \mathbb{R}$,

$$\|v\|_b := \operatorname*{ess\,sup}_{(c,x,t)} \frac{|v(c,x,t)|}{b(c,x,t)},$$

where we set $\frac{0}{0} := 0$, and we denote by $\mathbb{B}_b$ the Banach space

$$\mathbb{B}_b := \{v : [0,\infty) \times \mathbb{R} \times [0,T] \to [0,\infty) : v \text{ is measurable and } \|v\|_b < \infty\}\,.$$

Theorem 3.4 (Verification Theorem). *Let a piecewise deterministic Markov decision process be given with a bounding function b, and $\mathbb{E}\big(|b(C_T, X_T^\theta)|\big|C_0 = c, X_0 = x\big) < \infty$ for all θ, x. Suppose that $v \in C^{0,1,1}([0,\infty) \times \mathbb{R} \times [0,T]) \cap \mathbb{B}_b$ is a solution of the HJB equation and that f^* is a maximizer of the HJB equation and defines a state process (X_t^*).*

Then $v = V$ and $\theta^* = f^*(X_{t-}^*)$ is an optimal Markov policy (in feedback form).

Proof. This can be proved like [BR11, Theorem 8.2.8]. □

Remark 3.5. In the statement of Theorem 8.2.8 in [BR11] there is another condition required, namely that $\alpha_b < 1$ for a number α_b depending on b, Q and the arbitrary γ from Definition 3.3. But it is shown in [BR10] that for finite horizon problems γ can always be chosen large enough to satisfy $\alpha_b < 1$.

For the sake of brevity we fix k for the remainder of this subsection and we suppress the dependence of V on k.

The generator of (C_t, X_t^θ) is

$$\begin{aligned}\mathcal{A}^\theta(v)(c,x) = {} & q(\theta)\lambda\mathbb{E}(v(c+Y, x-Y) - v(c,x)) \\ & + (M - q(\theta))\lambda\mathbb{E}(v(c+Y,x) - v(c,x))\,,\end{aligned}$$

for $v : [0,\infty)\times\mathbb{R} \longrightarrow \mathbb{R}$ bounded and measurable. With this, the HJB equation for our problem is

$$0 = \sup_\theta \Big(V_t(c,x,t) + V_x(c,x,t)a(1+\theta_t)q(\theta_t) + \mathcal{A}V(c,x,t)\Big) \tag{6}$$

We introduce the shorthand notations

$$\begin{aligned}\hat{V}(c,x,s) &:= \lambda\mathbb{E}\big(V(c+Y,x,s) - V(c,x,s)\big) \\ \bar{V}(c,x,s) &:= \lambda\mathbb{E}\big(V(c+Y,x-Y,s) - V(c+Y,x,s)\big)\end{aligned}$$

Reordering of terms gives the compact form

$$\begin{aligned} V_t + M\hat{V} + \sup_{\alpha\in[0,m]}[q(\alpha)(a(1+\alpha)V_x + \bar{V})] &= 0 \\ V(c,x,T) &= u(x+k\psi(c))\,. \end{aligned}\tag{7}$$

We note that if $V \in C^{0,1,1}([0,\infty)\times\mathbb{R}\times[0,T]) \cap \mathbb{B}_b$, then $V_x > 0$ since u is strictly increasing and concave, so that the HJB equation can be written as

$$\begin{aligned} V_t + M\hat{V} + V_x \sup_{\alpha\in[0,m]}[q(\alpha)(a(1+\alpha) + \tfrac{\bar{V}}{V_x})] &= 0 \\ V(c,x,T) &= u(x+k\psi(c))\,. \end{aligned}\tag{8}$$

For all $z \in \mathbb{R}$ the function $\alpha \mapsto q(\alpha)(a(1+\alpha)+b)$ is continuous on $[0,m]$ and therefore attains its maximum. Define the function

$$\mu(z) := \max\{q(\alpha)(a(1+\alpha)+z) : \alpha \in [0,m]\}\,. \tag{9}$$

With this we can write down the following backward equation in V,

$$\begin{aligned} V_t(c,x,t) + M\hat{V}(c,x,t) + V_x(c,x,t)\mu\left(\tfrac{\bar{V}(c,x,t)}{V_x(c,x,t)}\right) &= 0 \\ V(c,x,T) &= u(x+k\psi(c))\,. \end{aligned}\tag{10}$$

Assumption 3.6. From now on, we restrict our considerations to exponential utility, $u(x) = -e^{-\eta x}$.

We make the usual ansatz $V(c,x,t) = u(x)e^{-\eta W(c,t)}$, such that

$$\begin{aligned} V_t(c,x,t) &= -\eta W_t(c,t)V(c,x,t) \\ V_x(c,x,t) &= -\eta V(c,x,t) \\ \hat{V}(c,x,t) &= V(c,x,t)\lambda\mathbb{E}(e^{-\eta(W(c+Y,t)-W(c,t))} - 1) \\ \bar{V}(c,x,t) &= V(c,x,t)\lambda\mathbb{E}((e^{\eta Y} - 1)e^{-\eta(W(c+Y,t)-W(c,t))})\,. \end{aligned}$$

We introduce similar short-hand notations as before:

$$\begin{aligned}\hat{W}(c,t) &:= -\frac{1}{\eta}\lambda\mathbb{E}(e^{-\eta(W(c+Y,t)-W(c,t))}-1)\\ \bar{W}(c,t) &:= -\frac{1}{\eta}\lambda\mathbb{E}((e^{\eta Y}-1)e^{-\eta(W(c+Y,t)-W(c,t))}).\end{aligned}$$

Substituting into the backward equation (10) for V and dividing by $-\eta V$ gives us a backward equation for W:

$$\begin{aligned}W_t(c,t)+M\hat{W}(c,t)+\mu(\bar{W}(c,t)) &= 0\\ W(c,T) &= k\psi(c)\end{aligned} \tag{11}$$

Note that since $Y \geq 0$ we have $e^{\eta Y}-1 \geq 0$ so that $\bar{W}(c,t) \leq 0$ always, with strict inequality if $\mathbb{P}(Y>0)>0$. As a consequence, the argument of μ will always be negative.

Note further that in order to have finite-valued $\bar{W}$ we need to make the following assumption.

Assumption 3.7. We assume $\mathbb{E}(e^{\eta Y})<\infty$.

If we can show that the backward equation (11) has a bounded and continuous solution which is differentiable with respect to the time component, and if we can present a maximizer for this solution, then we are done. We defer those proofs to Section 3.6.

For the time being we assume the existence of W and V, and see what we can do with it.

3.2. **Utility indifference price.** We can now – provided we can solve the corresponding backward equation (11) – compute the utility indifference price of a derivative with continuous and bounded payoff ψ. At time t the maximum expected terminal utility of terminal wealth if the derivative is bought at price p is given by

$$\begin{aligned}V(x-p,c,t,k) &= u(x-p)\exp(-\eta W(c,t,k))\\ &= u(x)\exp(\eta p)\exp(-\eta W(c,t,k)),\end{aligned}$$

where $W(.,.,k)$ is the solution to the backward equation (11). With no derivative bought, the maximum expected terminal utility of terminal wealth at time t is given by

$$V(c,x,t)=u(x)\exp(-\eta W(c,t,0)).$$

Therefore equation (4) takes on the following simple form:

$$p=p(c,t,k)=W(c,t,k)-W(c,t,0). \tag{12}$$

It should be noted that, since $W(c,T,0)\equiv 0$, the expression $W(c,T,0)$ does not depend on c. It follows from the equation (11) for W that also $W(c,t,0)$ for $t<T$ does not depend on c. $W(.,.,0)$ therefore may be computed as the solution of a one-dimensional ordinary differential equation: Let W^0 denote the solution to

$$\begin{aligned}W^0_t(t)+M\hat{W}^0(t)+\mu(\bar{W}^0(t)) &= 0\\ W^0(T) &= 0.\end{aligned} \tag{13}$$

Then $W(c,t,0) = W^0(t)$ for all c, t.

Lemma 3.8. $W^0(t) = \mu(-\frac{1}{\eta}\lambda\mathbb{E}(e^{\eta Y} - 1))(T-t)$.

Proof. Indeed, we see that if $W^0(t) = \kappa(T-t)$ for some constant κ, we have $\hat{W}^0(t) = 0$ and $\bar{W}^0(t) = \frac{1}{\eta}\lambda\mathbb{E}(e^{\eta Y} - 1)$ for all $t \leq T$. Therefore, $W^0(t)$ solves

$$\begin{aligned} W^0_t(t) + M\hat{W}^0(t) + \mu(\bar{W}^0(t)) &= 0 \\ W^0(T) &= 0 \end{aligned}$$

iff $\kappa = \mu(-\frac{1}{\eta}\lambda\mathbb{E}(e^{\eta Y} - 1))$.

The uniqueness of this solution follows from the general existence and uniqueness theorem in Section 3.6, Theorem 3.12. $\square$

Using this lemma we can write down a backward equation for p: Since from equation (12) and Lemma 3.8

$$p = p(c,t,k) = W(c,t,k) - \kappa(T-t) \tag{14}$$

with $\kappa = \mu(-\frac{1}{\eta}\lambda\mathbb{E}(e^{\eta Y} - 1))$, we get from (11)

$$\begin{aligned} p_t(c,t,k) - \kappa + M\hat{p}(c,t,k) + \mu(\bar{p}(c,t,k)) &= 0 \\ p(c,T,k) &= k\psi(c)\,, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \hat{p}(c,t,k) &:= -\tfrac{1}{\eta}\lambda\mathbb{E}(e^{-\eta(p(t,c+Y,k)-p(c,t,k))} - 1) \\ \bar{p} &:= -\tfrac{1}{\eta}\lambda\mathbb{E}((e^{\eta Y} - 1)e^{-\eta(p(t,c+Y,k)-p(c,t,k))})\,. \end{aligned} \tag{16}$$

We now look at another aspect of the backward equation in the special case where the payoff is such that for some positive L we have $\psi(c) = A$ for all $c > L$. This is certainly satisfied for the aforementioned examples of spread option and CAT-bond and indeed for most reasonable bounded payoff functions.

Under this assumption we have for $c \geq L$ that $\hat{p}(c,t) = 0$ and $\bar{p}(c,t) = -\frac{1}{\eta}\lambda\mathbb{E}(e^{\eta Y} - 1)$ for $c > L$. Therefore we get from (15) and the definition of κ

$$p_t(c,t) = 0$$

for $c \geq L$, and since $p(c,T) = A$,

$$p(c,t) = A$$

for $c \geq L$. This means that if ψ is constant above a cutoff level L, then we have to compute W for $c \in [0, L]$ only. This obviously simplifies the numerics. We want to stress however, that all the theoretical results hold without the above assumption and the numerics can deal with this case quite analog to many other pricing models where the solution on an unbounded interval is approximated by a function on a compact interval.

3.3. **Linear demand.** We now consider the particularly simple case where q is linear on the interval $[0, m]$, i.e.

$$q(\theta) = M \min(1, \max(1 - \frac{\theta}{m}, 0)) .$$

Here μ and can simply be calculated:

$$\mu(z) = \begin{cases} 0 & z \leq -a(m+1) \\ M(a+z) & z \geq a(m-1) \\ \frac{M}{4am}(a(1+m)+z)^2 & \text{else} \end{cases}$$

and the maximum is attained in

$$\gamma(z) := \begin{cases} m & z \leq -a(m+1) \\ 0 & z \geq a(m-1) \\ \frac{a(m-1)-z}{2a} & \text{else} . \end{cases}$$

The simple proof is left to the reader.

3.4. **Demand functions with unique optimal risk loading.** In our general derivation the only requirements on the demand function q were that it be continuous, non-increasing with

$$q(\theta) = \begin{cases} M & \text{if } \theta \leq 0 \\ 0 & \text{if } \theta \geq m . \end{cases} \tag{17}$$

In general such a function q will lead to more than one optimal strategy. Though the value function does not depend on the particular choice of the optimal strategy, multiple optimal strategies generate practical difficulties, for example for numerical computation of the optimal strategy/value functions.

Linear demand is obviously not the only example that allows for exact and unique computation of the optimal strategy. Let us consider the class of functions q which satisfy (17), are twice continuously differentiable on $(0, m)$, have negative derivative on $(0, m)$ and for which

$$f_q(\alpha, z) := q(\alpha)(a(1+\alpha)+z)$$

has a unique maximum in $(-\infty, m]$ for all b.

From our assumptions on q we have $q(\alpha) \geq 0$ so that $f_q(\alpha, z) < 0$ iff $a(1+\alpha)+z < 0$. Since $f_q(m, z) = 0$, we therefore know that the optimal α, if it exists, must satisfy $a(1+\alpha)+z > 0$, i.e. $\alpha > -1 - \frac{z}{a}$.

We are therefore interested in demand functions q for which $f_q(., z)$ has a unique maximum in $(-1 - \frac{z}{a}, m]$ if $-1 - \frac{z}{a} < m$. (For $-1 - \frac{z}{a} \geq m$ the function $f_q(., z)$ attains its maximum in m.)

A sufficient condition for this is that $\alpha \mapsto f_q(\alpha, z)$ is strictly concave on $[-1 - \frac{z}{a}, m]$.

Theorem 3.9. *Let q be of the form*

$$q(\theta) = \begin{cases} M & \text{if} \quad \theta \leq 0 \\ 0 & \text{if} \quad \theta \geq m \\ M - \int_0^\alpha e^{-\frac{2\xi}{1+m}} H(\xi) d\xi & \text{else} \end{cases}$$

for some function $H : [0, m] \to \mathbb{R}$ satisfying

(i) *H is differentiable on $(0, m)$;*

(ii) $H' > 0$ *on* $(0, m)$;
(iii) $H > 0$ *on* $(0, m)$;
(iv) $\int_0^m e^{-\frac{2\xi}{1+m}} H(\xi) d\xi = M$.

Then $f_q(., z)$ *is strictly concave on* $[-1 - \frac{z}{a}, m]$.

Proof.

$$\frac{\partial^2}{\partial \alpha^2} f_q(\alpha, z) = q''(\alpha)(a(1+\alpha)+z) + 2q'(\alpha)a\,,$$

which is negative for $\alpha \in (-1 - \frac{z}{a}, m)$ iff

$$\frac{q''(\alpha)}{q'(\alpha)} > -\frac{2a}{a(1+\alpha)+z}\,.$$

Since $z \leq 0$, the right hand side is always greater or equal to $-\frac{2a}{a(1+\alpha)} = -\frac{2}{(1+\alpha)} \geq -\frac{2}{(1+m)}$.

We have therefore shown that if

$$\frac{q''(\alpha)}{q'(\alpha)} = -\frac{2}{1+m} + h_1(\alpha) \tag{18}$$

for some continuous function $h_1 : [0, m] \to \mathbb{R}$ which is positive on $(0, m)$, then $f_q''(\alpha, z) < 0$ for all $\alpha \in (-1 - \frac{z}{a}, m)$, $z \leq 0$. But

$$\begin{array}{rrcl} & \frac{q''(\alpha)}{q'(\alpha)} & = & -\frac{2}{1+m} + h_1(\alpha) \\ \iff & \frac{d}{d\alpha}\log(|q'(\alpha)|) & = & -\frac{2}{1+m} + h_1(\alpha) \\ \iff & \log(|q'(\alpha)|) & = & -\frac{2\alpha}{1+m} + h_2(\alpha) \\ \iff & q'(\alpha) & = & -\exp\left(-\frac{2\alpha}{1+m} + h_2(\alpha)\right) \\ \iff & q'(\alpha) & = & -\exp\left(-\frac{2\alpha}{1+m}\right) H(\alpha) \end{array}$$

where h_2 is a primitive function of h_1, i.e. $h_2' = h_1$ and $H(\alpha) = \exp(h_2(\alpha))$. H is a positive, continuously differentiable function with $H' > 0$. So if with this H we define

$$q(\alpha) = M - \int_0^\alpha \exp\left(-\frac{2\xi}{1+m}\right) H(\xi) d\xi$$

we have $q(0) = M$ and if we further have

$$\int_0^m \exp\left(-\frac{2\xi}{1+m}\right) H(\xi) d\xi = M\,,$$

then $q(m) = 0$. □

Note that we recover linear demand for $H(\xi) = \frac{M}{m} e^{\frac{2\xi}{1+m}}$. Other simple examples are arrived at by using a function H of the form $H(\xi) = c\,P(\xi) e^{\frac{2\xi}{1+m}}$, where P is some polynomial which satisfies $P(\xi) > 0$ and $P'(\xi) > 0$ for $\xi \in (0, m)$ and

$$c = M \left(\int_0^m P(\xi) d\xi \right)^{-1}.$$

Other noteworthy examples are provided by $q(\alpha) = M(1 - (\frac{\alpha}{m})^\nu)$ on $[0, m]$ with $\nu > 0$.

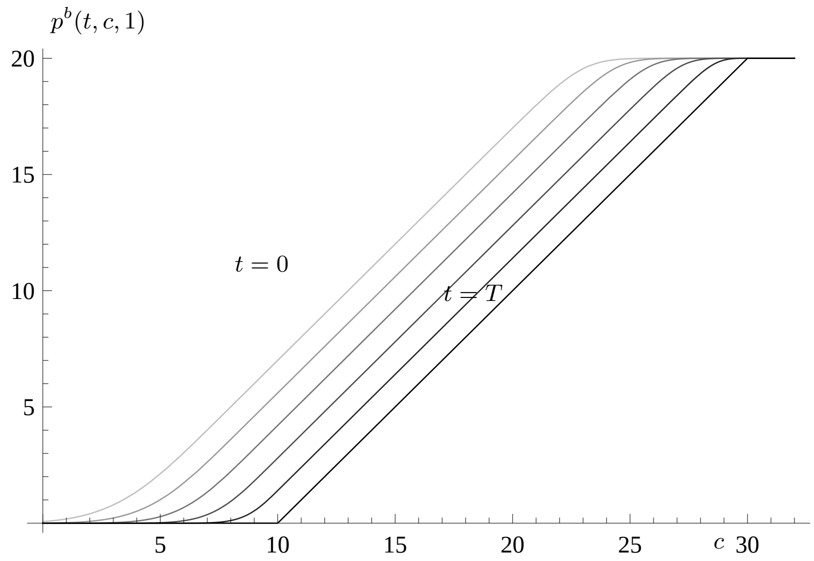

FIGURE 1. The buyer's price $p^b(.,t)$ for various values of $t \in [0,T]$. Lighter shades of gray correspond to earlier times.

3.5. **Numerical example.** In this section we consider a numerical example for the model proposed earlier. All further illustrations will refer to this setup.

We concentrate on the special case of linear demand from Section 3.3 and take the following values for the problem: $T = \frac{1}{4}$, $\lambda = 0.01$, $M = 10^4$, $m = 2$. The claim sizes Y_i are distributed on $\{\delta, 2\delta, 3\delta, 4\delta, 5\delta\}$ with $\delta = 10^5$ and corresponding probabilities $\frac{1}{8}, \frac{3}{8}, \frac{2}{8}, \frac{1}{8}, \frac{1}{8}$. The risk aversion coefficient is $\eta = 10^{-6}$.

We consider the payoff $\psi(c) = \max(0, \min(c - K, L - K))$, where $K = 10^7$, $L = 3 \cdot 10^7$. We therefore have $\psi(c) = A$ for $c > L$ with $A = 2 \cdot 10^7$. Note that in our setup the PIDEs (11) and (20) become ordinary differential equations in $\mathbb{R}^n$, where $n = \frac{A}{\delta} + 1$. This has the consequence that this example can be computed rather efficiently.

Figure 1 shows the price $p^b(.,t)$ of the derivative as a function of C_t. The darkest line shows the price at expiry, which is equal to the payoff, the lightest line shows the price at time $t = 0$. Both axes are million units of currency.

As is to be expected, the price of the derivative is a smoothed version of the payoff shifted to the left. This shift is due to the fact that the claims process is non-decreasing with time.

Figure 2 shows the risk loading θ_t as a function of C_t. The darkest line shows the risk-loading close to expiry, the lightest line shows the risk-loading at time $t = 0$. The x-axis is in million units of currency. We see that for some parameters (e.g. $t = 0$, $C_t = 15 \cdot 10^6$), due to the presence of the derivative the risk loading is pushed down roughly from 1.09 to 0.93. That means that the derivative makes the insurance more than 10 percent cheaper. Obviously this effect vanishes for $C_t > L$, in which case the derivative corresponds to a

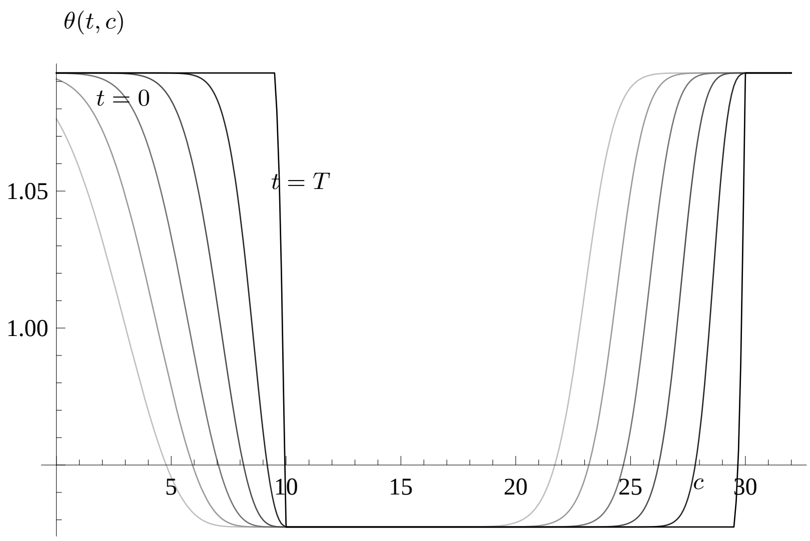

FIGURE 2. Here we plot the risk loading $\theta(c,t)$ as a function of c. Lighter shades of gray correspond to earlier times.

deterministic payment, such that the risk loading reverts to the one without a derivative present.

For C_t far below the "strike" K the risk loading is also higher, which can be explained by the relatively high probability for the derivative to have zero payoff, such that the risk loading is close to that without a derivative present.

3.6. **Verification.** Now we want to convince ourselves that there exists a solution to (11), with a corresponding maximizer and that the conditions of Theorem 3.4 are satisfied.

Lemma 3.10. $b(c,x) := \exp(\eta x)$ *is a bounding function for our piecewise deterministic Markov decision model with bounded payoff function* ψ.

Proof. (i) $u(c,x) = -\exp(-\eta(x+\psi(c))$ such that $|u(c,x)| = \exp(-\eta(x+\psi(c)) \leq \exp(\eta\|\psi\|_\infty)b(c,x)$.
(ii)

$$\begin{aligned}\int b(c,x)Q(dc\times dx|c,x,\theta) &= \int \exp(\eta x)Q(dc\times dx|c,x,\theta)\\ &= q(\theta)\lambda\mathbb{E}(\exp(\eta(x-Y))) + (M-q(\theta))\lambda\mathbb{E}(\exp(\eta x)))\\ &\leq b(c,x)\sup_\theta\left(q(\theta)\lambda\mathbb{E}(\exp(-\eta Y)) + (M-q(\theta))\lambda\right)\\ &\leq \lambda M b(c,x)\,.\end{aligned}$$

(iii) Let $\zeta := \sup_{y\in[0,m]} aq(y)(1+y)$. Then

$$\begin{aligned}b(c,x+\int_0^T\int_0^m aq(y)(1+y)\alpha_s(dy)ds) &= \exp\left(\eta x + \eta\int_0^T\int_0^m aq(y)(1+y)\alpha_s(dy)ds\right)\\ &\leq \exp(\eta T\zeta)b(c,x)\,.\end{aligned}$$

□

Recall the function

$$\mu(z) := \max\{q(\alpha)(a(1+\alpha)+z) : \alpha \in [0,m]\}$$

and define the multivalued correspondence

$$\Gamma(z) := \{\alpha \in [0,m] : q(\alpha)(a(1+\alpha)+z) = \mu(z)\}\,.$$

Lemma 3.11. (1) *μ is a convex function on $\mathbb{R}$.*
(2) *μ is Lipschitz-continuous on compact sub-intervals of $\mathbb{R}$.*

Proof. From our assumptions on q it follows that there exists a continuous inverse q^{-1} to q on $[0,M]$. First note that

$$\begin{aligned}\mu(z) &= \max\{q(\alpha)(a(1+\alpha)+z) : \alpha \in [0,m]\}\\ &= a\max\{q(\alpha)\alpha + q(\alpha)(z/a+1) : \alpha \in [0,m]\}\\ &= a\max\{\beta q^{-1}(\beta) + \beta(z/a+1) : \beta \in [0,M]\}\\ &= a\max\{-f(\beta) + \beta(z/a+1) : \beta \in [0,M]\}\\ &= af^*(z/a+1)\,,\end{aligned}$$

where

$$f(\beta) := \begin{cases} -\beta q^{-1}(\beta) & \beta \in [0,M] \\ \infty & \beta \in \mathbb{R}\setminus[0,M] \end{cases}$$

and f^* denotes the convex conjugate of f, cf. 12 in [Roc70]. f^* is a convex function on all of $\mathbb{R}$, see Theorem 12.2 in [Roc70]. f^* is real-valued on all of $\mathbb{R}$ since

$$\beta\beta^* - f(\beta) = \begin{cases} \beta\beta^* + \beta q^{-1}(\beta) & \beta \in [0,M] \\ -\infty & \beta \in \mathbb{R}\setminus[0,M] \end{cases}$$

such that

$$\sup_{\beta\in\mathbb{R}}(\beta\beta^* - f(\beta)) = \max_{\beta\in[0,M]}(\beta\beta^* + \beta q^{-1}(\beta)) \in \mathbb{R}\,.$$

Therefore f^* is Lipschitz on compact intervals, cf. Theorem 10.4 in [Roc70], and so is μ. □

Theorem 3.12. *Let ψ be a continuous and bounded function on $\mathbb{R}$ and let $\mathbb{E}(e^{\eta Y}) < \infty$. Then the backward equation (11) for W has a unique solution.*
Moreover, the solution is bounded.

Proof. Consider the Banach space $C_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$. The backward equation (11) is just an initial value problem for a $C_b(\mathbb{R})$-valued function,

$$\begin{aligned} w'(t) &= G(t,w(t)) \\ w(T) &= \psi\,, \end{aligned}$$

where $G(t,w)(c) = -M\hat{w}(c,t) - \mu(\bar{w}(c,t))$. Using Lemma 3.11 it is readily shown that G satisfies a local Lipschitz condition in the second variable with respect to the sup-norm. Therefore the classical Picard-Lindelöf theorem on existence and uniqueness of solutions of ODEs gives us the unique solution w

to the initial value problem. Since each $w(.)$ is bounded and w is continuous, we have that the function

$$\begin{array}{ccc} [0,T]\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ (c,t) & \longmapsto & w(t)(c) \end{array}$$

is bounded. □

Lemma 3.13. *There is a measurable function $\gamma:\mathbb{R}\longrightarrow[0,m]$ such that*

$$q(\gamma(z))(a(1+\gamma(z))+z)=\mu(z)$$

for all $z\in\mathbb{R}$.

Proof. Recall the correspondence Γ which maps $z\in\mathbb{R}$ to the compact set of all $\alpha\in[0,m]$ which maximize $q(\alpha)(a(1+\alpha)+z)$. Berge's Theorem of the Maximum ([Ber97] p.116), states that Γ is upper hemi-continuous, such that its graph is closed. The graph $\mathcal{G}_\Gamma$ is therefore a closed subset of $\mathbb{R}\times[0,m]$ and the projection of $\mathcal{G}_\Gamma$ to $\mathbb{R}$ is $\mathbb{R}$. From von Neumann's Measurable Choice Theorem (see [Dx08] Appendix V) it therefore follows that there exists a measurable function $\gamma:\mathbb{R}\longrightarrow[0,m]$ such that $q(\gamma(z))(a(1+\gamma(z))+z)=\mu(z)$ for all $z\in\mathbb{R}$. □

Theorem 3.12 and Lemma 3.13 together show that the assumptions of the Verification Theorem 3.4 are satisfied.

4. Certainty equivalence price and limit prices

In the preceding section we computed the utility indifference price. Having done this we might ask whether there can actually be a trade, that is whether it may occur that $p(c,t,k)=p^b(c,t,k)\geq p^s(c,t,k)=-p(c,t,-k)$. This does not seem to be the case too often. All of our numerical examples show that $p^b(k)<p^s(k)$. This is in line with, e.g., the findings by Takino [Tak07], who computes indifference prices of European claims in a stochastic volatility model with partial information.

4.1. **Certainty equivalence price.**

Another interesting question is what an investor is willing to charge for the derivative if she cannot hedge the derivative. The hedging strategy we proposed earlier can only be realized by an insurance company. It is not unreasonable to assume that the counterpart in such a deal is *not* an insurance company. In that case the utility indifference price of the buyer coincides with the certainty equivalence price.

It is also reasonably to assume that on the seller's side the derivative is split up in the way mentioned above, i.e. each seller sells only the N-th part of the whole derivative. The example we have in mind is that of a CAT-bond which is denominated into N units.

Let us denote the certainty equivalence price for the seller by $\pi^s(c,t,k)$, i.e. the solution to

$$\mathbb{E}\big(-\exp(-\beta y_0)|C_t=c\big)=\mathbb{E}\Big(-\exp\big(-\beta(y_0+\pi^s(c,t,k)-k\psi(C_T))\big)|C_t=c\Big),$$

where we have assumed exponential utility with coefficient of risk aversion β. That is

$$\pi^s(c,t,k) = \frac{1}{\beta}\log\left(\mathbb{E}(\exp(\beta k\psi(C_T))|C_t = c)\right).$$

π^s may be computed either directly using the distribution of C_T or using a backward equation which can be derived similarly to the backward equation for W:

$$\begin{aligned} \pi^s_t(c,t,k) + M\tfrac{\lambda}{\beta}\mathbb{E}\left(e^{\beta\pi^s(c+Y,t,k)} - e^{\beta\pi^s(c,t,k)}\right) &= 0 \\ \pi^s(c,T,k) &= k\psi(c)\,. \end{aligned} \tag{19}$$

Therefore the derivative can only be sold in denomination N, between N sellers without the opportunity to hedge and a buyer with the opportunity to hedge, if

$$p^b(c,t,1) \geq N\pi^s(c,t,1/N) = \frac{N}{\beta}\log(\mathbb{E}(\exp(\frac{\beta}{N}\psi(C_T))|C_t = c))$$

for some N. For bounded ψ and $N \to \infty$ we have $\exp(\frac{\beta}{N}\psi(C_T)) \approx 1 + \frac{\beta}{N}\psi(C_T)$ and therefore $\mathbb{E}(\exp(\frac{\beta}{N}\psi(C_T))|C_t = c) \approx 1 + \frac{\beta}{N}\mathbb{E}(\psi(C_T)|C_t = c)$ and further $\frac{N}{\beta}\log\mathbb{E}(\exp(\frac{\beta}{N}\psi(C_T))|C_t = c) \approx \mathbb{E}(\psi(C_T)|C_t = c)$. That is

$$\lim_{N\to\infty} N\pi^s(c,t,1/N) = \mathbb{E}(\psi(C_T)|C_t = c)\,.$$

Define

$$\pi^0(c,t) := \mathbb{E}(\psi(C_T)|C_t = c)\,,$$

then v can be computed directly or via the backward equation

$$\begin{aligned} v_t(c,t) + M\lambda\mathbb{E}(\pi^0(c+Y,t) - \pi^0(c,t)) &= 0 \\ \pi^0(c,T) &= \psi(c)\,, \end{aligned} \tag{20}$$

and gives the limit of the buyer's price for $N \to \infty$.

In our numerical example we have

$$p^b(c,t,1) > \lim_{N\to\infty} N\pi^s(c,t,\frac{1}{N}) = \pi^0(c,t)\,,$$

that is a deal could be stricken for sufficiently large N, *provided that the sellers of the derivative are not able to hedge the derivative.* The corresponding difference between utility indifference buyer's price and π^0, the denomination limit of π^s, is shown in Figure 3.

4.2. **Risk-neutral limit.** Another interesting quantity is the risk-neutral limit of the indifference price, that is

$$p_0(c,t,k) := \lim_{\eta\to 0} p_\eta(c,t,k)\,,$$

where p_η is the utility indifference price corresponding to risk aversion η. The risk-neutral limit price has been considered, for example, in [RK00, DGR02, Bec01, Bec03], and it is of some interest in that it gives a linear pricing rule which nevertheless is related to the non-linear utility indifference pricing rule.

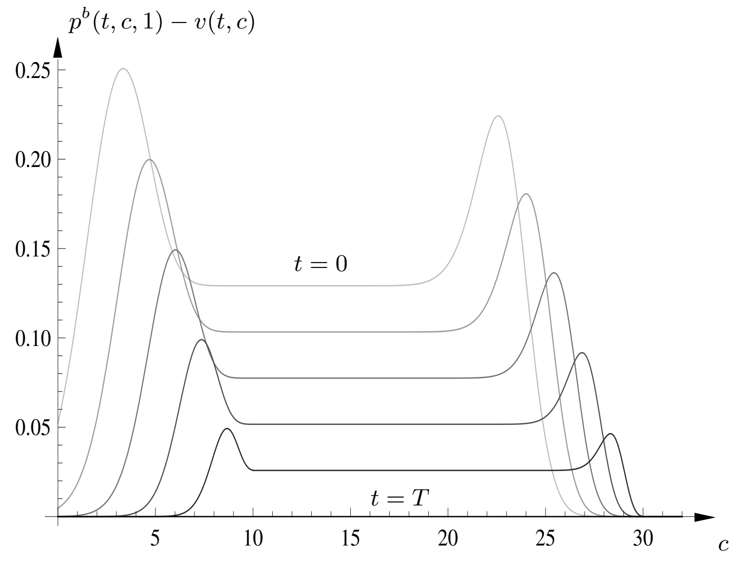

FIGURE 3. Here we plot $p^b - \pi^0$ as a function of c. Lighter shades of gray correspond to earlier times. Note that the difference is always non-negative, which implies that the derivative can actually be traded.

It is not hard to see that our optimization problem (5) is meaningful even for $\eta = 0$, that is if one takes linear utility $u(x) = x$. It turns out that $p_0(c,t,k) = k\pi^0(c,t)$ where π^0 is the same as in the preceding subsection.

5. Conclusion and open questions

In our study, we have modeled the industrial loss index by a compound Poisson process and showed that the insurer can control her wealth process by adjusting her portfolio via choosing the risk loading. Our study contributes to the insurance theory by showing that by issuing CAT bonds and offering catastrophe coverage the net expected income of the insurance company remains the same while the insurer can lower the premium charge.

The study has a greater significance to low-income countries where natural disasters often exceeds the resources of internal and external sources of relief funding: Using our strategy, the insurance company in the low-income country can sell CAT bond to some (ethical) investors and offer affordable insurance services against risk of low-probability, high-loss events to the needy/poor vulnerable customers.

We have discussed to role of the ability to hedge. We have found that in the natural situation where the seller (or the sellers) of the derivative is not an insurance company and therefore cannot hedge the derivative via her portfolio, then the derivative can actually be bought.

For future research, the following extensions or generalizations of the problem are of interest:

- N could be a doubly stochastic process, such that its intensity varies over time. This would allow for a more realistic modeling of catastrophic events: One could have "normal" times, where claims arrive at a low rate and "catastrophe" times, where claims arrive at a very high rate. One would then probably restrict the policies of the insurance company in a way that does not permit changing the risk loading during catastrophe times.
- Alternatively, one could model catastrophes as events where several claims happen at the same time and where the insurance get to pay a random number of claims according to their fraction of the total portfolio.
- It would be interesting to allow for some lag for the adjustment of the demand to a changed risk loading.
- The Y_k's and U_k's could be made dependent. More specifically, the parameters of Y_k could be a function of U_k. This is reasonable, since the different clients are likely to have different claim distributions. In that setup, the policies of the insurance company would be more complicated objects, the risk loading would also depend on the parameters of the claim distributions.

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