CYCLE INDICES OF $PGL(2,q)$ ACTING ON THE COSETS OF ITS SUBGROUPS

S.K. Rotich

Department of Mathematics and Statistics
Machakos University
P.O. Box 136,
Machakos-Kenya
Corresponding author
E-mail: starotich@gmail.com

ABSTRACT

In this paper, we mainly investigate the action of $PGL(2,q)$ on the cosets some of its subgroups namely; C_{q-1}, C_{q+1} and P_q. Corresponding to each action the disjoint cycle structures and cycle index formulas is determined.

***Key words**: Disjoint cycle structures, Cycle index formula.*

1 Introduction

Projective General Linear Group

The Projective General Linear Group $PGL(2,q)$ over a finite field $GF(q)$, where $q = p^f$, p a prime number and f a natural number, is a group consisting of all linear fractional transformations of the form;

$$x \rightarrow \frac{ax+b}{cx+d} \ ,$$

where $x \in PG(1,q) = GF(q) \cup \{\infty\}$, the projective line, $a, b, c, d \in GF(q)$ and

$$ad - bc \neq 0.$$

It is the factor group of the general linear group by its centre and it is of order $q(q^2 - 1)$.

(Dickson [1])

$PGL(2, q)$is the union of three conjugacy classes of subgroups each of which intersects with each other at the identity subgroup, and these are;

a) Commutative subgroups of order q

$PGL(2, q)$has commutative subgroups of order q. We denote it by P_q.

Non- identity transformations in P_q have only∞ as a fixed point in $PG(1, q)$. It is of the form;

$$P_q = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in GF(q) \right\},$$

The normalizer of P_q in $PGL(2, q)$has transformations of the form;

$$S = \begin{pmatrix} c & e \\ 0 & d \end{pmatrix} \text{ } where\ e \in GF(q)\ and\ c, d \in GF(q)^*.$$

Thus ,

$$\left| N_{PGL(2,q)}(P_q) \right| = q(q - 1).$$

Therefore the number of subgroups of $PGL(2, q)$conjugate to P_q is $q + 1$.

These $q + 1$ subgroups have no transformation in common except the identity. All the conjugate subgroups of order q contain $q^2 - 1$ distinct non identity transformations of order p.

b) Cyclic subgroups of order $q-1$

$PGL(2,q)$also contains a cyclic subgroup C_{q-1} of order $q-1$. Each of its non identity transformations fixes 0 and ∞. It is of the form;

$$C_{q-1} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in GF(q)^* \right\}.$$

The normalizer of C_{q-1} is generated by the transformations T and W

Where; $T = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$,

and h is a primitive element in $GF(q)$ and

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus ,

$\left| N_{PGL(2,q)}(C_{q-1}) \right| = 2(q-1)$.

Therefore the number of subgroups of $PGL(2,q)$ conjugate to $C_{q-1}$is $\frac{1}{2}q(q+1)$. These conjugate cyclic subgroups have no transformation in common except the identity and thus contain$\frac{1}{2}q(q+1)(q-2)$ non identity transformations

c) **Cyclic subgroups of order $q+1$**

Another subgroup of $PGL(2,q)$ is the cyclic subgroup C_{q+1} of order $q+1$. This consists of elliptic transformations. All the non identity elements of the cyclic subgroup of order $q+1$ fixes no element. The normalizer of this subgroup is a dihedral subgroup of order $2(q+1)$.

Therefore the number of subgroups of $PGL(2,q)$ conjugate to this cyclic subgroup of order $q+1$ is $\frac{1}{2}q(q-1)$.

Since the $\frac{1}{2}q(q-1)$ conjugate subgroups have only identity in common so, they contain $\frac{1}{2}q^2(q-1)$ non identity transformations.(Huppert [2])

1.1 Theorem

Let G be a finite transitive permutation group acting on the right cosets of its subgroup H. if $g \epsilon G$ and $|G:H| = n$ then,

$$\frac{\pi(g)}{n} = \frac{|C^g \cap H|}{|C^g|}.$$

(Kamuti [4])

1.2 Lemma

Let g be a permutation with cycle type $(\propto_1, \propto_2, \propto_3, \dots\dots\dots, \propto_n)$ then,

a) The number $\pi(g^l)$ of 1-cycle in g^l is $\sum_{i|l} i\alpha_i$

b) $\alpha_i = \frac{1}{i}\sum_{i|l} \pi(g^{l/i})\mu(i)$, where μ is the mobius function.

(Hardy and Wright [3])

1.3 Lemma

If g is elliptic or hyperbolic of order greater than 2 or if g is parabolic, then the centralizer in *G* consists of all elliptic (respectively, hyperbolic, parabolic) elements with the same points set, together with the identity elements. On the other hand if g is elliptic or hyperbolic of order 2, then its centralizer is the dihedral group of order $2(q+1)$ or $2(q-1)$ respectively.

(Dickson [1])

1.4 Theorem

a) Let $\wp$ be the following set of subgroups of $G = PGL(2,q)$;

$$\wp = \{P_q^g, C_{q-1}^g, C_{q+1}^g | g \in G\}.$$

Then each non-identity elements of *G* is contained in exactly one group in $\wp$. (Thus the set $\wp$ form a partition of *G*.)

b) Let $\pi(g)$ be the number of fixed points of $g \in G$ on the $PG(1,q)$. If we define

$$\tau_i = \{g | g \in G, \pi(g) = i\};$$

then

$\tau_0 = \cup (c_{q+1} - I)^g, \tau_1 = \cup (P_q - I)^g, \tau_2 = \cup (c_{q-1} - I)^g$.(Kamuti [4])

1.5 Theorem

The cycle index of the regular representation of a cyclic group C_n is given by;

$$Z(C_n) = \frac{1}{n}\sum_{d|n} \varphi(d) t_d^{n/d},$$

whereφ is the Euler's Phi function and $t_{1,} t_{2,} \ldots \ldots \ldots, t_n$ are distinct (commuting) indeterminates.

(Redfield [5])

2.0 PERMUTATION REPRESENTATIONS OF $G = PGL(2, q)$

2.1 Representation of *G* on the cosets of $H = C_{q-1}$

G contains a cyclic subgroup *H* of order $q-1$whose every non-identity elements fixes two elements. If g is an element in *G*, we may want to find the disjoint cycle structures of the permutation g' induced by g on the cosets of *H*.Our computation will be carried out by each time taking an element $g$of order d in *G* from $\tau_{1,}\tau_2$ and τ_0 respectively.

To find the disjoint cycle structuresof $g \in G$we use Lemma 1.2(b) and thus we need to determine$|C^g|$,$|C^g \cap H$ and πg using Theorem 1.1 We easily obtain Cg using Lemma 1.3, but we need to distinguish between $d=2$ and $d > 2$. So if $d > 2$ then we have;

$$|C^g| = \frac{|G|}{|C(g)|},$$

where$|C(g)|$is the order of centralizer of g.

The values of $\pi(g)$are displayed in Table 2.1.1 below

Table 2.1.1: No. of fixed points of elements of *G* acting on the cosets of C_{q-1}

Case		$\|C^g\|$	$\|C^g \cap H\|$	$\pi(g)$
I	$g \in \tau_0 d > 2$	$q(q-1)$	0	0
	$d = 2$	$\frac{q(q-1)}{2}$	0	0
II	$g \in \tau_1\ d > 2$	$q^2 - 1$	0	0
	$d = 2$	$q^2 - 1$	0	0
III	$g \in \tau_2\ d > 2$	$q(q+1)$	2	2
	$d = 2$	$\frac{q(q+1)}{2}$	1	2

After obtaining $\pi(g)$, we proceed to calculate in details the disjoint cycles structures of elementsg' in this representation using Lemma 1.2(b).

Table 2.1.2: Disjoint cycle structures of elements of *G* on the cosets of $\boldsymbol{C_{q-1}}$

	τ_1		τ_0		τ_2	
Cycle length of $g^{'}$	1	p	1	d	1	d
No. of cycles	0	$p^{f-1}(q+1)$	0	$\frac{q(q+1)}{d}$		$2\frac{(q-1)(q+2)}{d}$

2.2 Representation of *G* on the cosets of $H = C_{q+1}$

To compute the disjoint cycle structures of $g^{'}$ we also need to determine $\pi(g)$ by using Theorem 1.1 and find α_l by applying Lemma 1.2(b). Before we obtain $\pi(g)$ we first need to determine $|C^g|$ and$|C^g \cap H|$. Since$|C^g|$ is the same as in Section 2.1, so we only need to find $|C^g \cap H|$. If $g\epsilon\tau_1$ and $\tau_2$then,

$$|C^g \cap H| = 0.$$

. If $g\epsilon\tau_0$ then,

$$|C^g \cap H| = \begin{cases} 1 \; if \; d = 2 \\ 2 \; if \; d \neq 2 \end{cases}.$$

This is because if$d = 2$ $\varphi(2) = 1$. So it has a single element of order 2. Each subgroup of *G* conjugate to$H$ has one element of order 2.

If $g\epsilon\tau_0$ and $d \neq 2$we have;

$$|C^g \cap H| = 2$$

The values of $\pi(g)$ are displayed in Table 2.2.1 below

Table 2.2.1: No. of fixed points of elements of *G* acting on the cosets of $\boldsymbol{C_{q+1}}$

Case		$\|C^g\|$	$\|C^g \cap H\|$	$\pi(g)$
I	$g \in \tau_0 d > 2$	$q(q-1)$	2	2
	$d = 2$	$\frac{q(q-1)}{2}$	1	2
II	$g \in \tau_1\ d > 2$	$q^2 - 1$	0	0
	$d = 2$	$q^2 - 1$	0	0
III	$g \in \tau_2\ d > 2$	$q(q+1)$	0	0
	$d = 2$	$\frac{q(q+1)}{2}$	0	0

After obtaining $\pi(g)$ we use the same approach as in Section 2.1 above to find the disjoint cycle structures of elements of *G* on the cosets of C_{q+1}.Therefore the results is as shown in Table 2.2.2 below.

Summary of results

Table 2.2.2: Disjoint Cycle structures of elements of *G* on the cosets of $\boldsymbol{C_{q+1}}$

	τ_1		τ_0		τ_2	
Cycle length of g'	1	p	1	d	1	d
No. of cycles	0	$p^{f-1}(q-1)$	2	$\frac{(q+1)(q-2)}{d}$	0	$\frac{q(q-1)}{d}$

2.3 Representations of G on the cosets of $H = P_q$

To obtain the disjoint cycle structures of g we use Theorem 1.1.9 to get $\pi(g)$. We compute α_l using Lemma 1.1.10(b). To find $\pi(g)$ we first need to obtain $|C^g|$and $|H \cap C^g|$. Since$|C^g|$ is the same as in Section 2.1, so we only need to determine $|H \cap C^g|$.

If $g \in \tau_0$ and $g \in \tau_2$,

$$|H \cap C^g| = 0.$$

If $g \in \tau_1$ then;

$$|H \cap C^g| = q - 1,$$

The values of $\boldsymbol{\pi(g)}$ are displayed in Table 2.3.1 below;

Table 2.3.1: No. of fixed points of elements of *G* on the cosets of $\boldsymbol{p_q}$

Case		$\|C^g\|$	$\|C^g \cap H\|$	$\pi(g)$
I	$g \in \tau_0$	$q(q-1)$	0	0
II	$g \in \tau_1$	$q^2 - 1$	$q - 1$	$q - 1$
III	$g \in \tau_2$	$q(q+1)$	0	0

We use similar approach to find the disjoint cycle structures of elements of *G* on the cosets of P_q as section 2.1.

Table 2.3.2: Disjoint Cycle structures of elements of *G* on the cosets of $\boldsymbol{p_q}$

	τ_1	τ_0	τ_2
Cycle length of g'	1 p	1 d	1 d
No. of cycles	$q-1$ $\frac{q(q-1)}{p}$	0 $\frac{(q^2-1)}{d}$	0 $\frac{(q^2-1)}{d}$

3.0 THE CYCLE INDEX FORMULAS FOR $G = PGL(2, q)$ ACTING ON THE COSETS OF ITS SUBGROUPS

After computing the disjoint cycle structures of elements G acting on the cosets of its subgroups, then we can now use them to find the cycle index formula for these representations.

In this section we shall determine some general formulas for finding the cycle indices for the representation of G on the cosets of its subgroups.

3.1 Cycle index of G acting on the cosets of $H = C_{q-1}$

Theorem

The cycle index of G on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{|G|/|H|} + (q^2-1)t_p^{p^{f-1}(q+1)} + \frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_1^2 t_d^{\frac{(q-1)(q+2)}{d}} + \frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_d^{\frac{q(q+1)}{d}}\right].$$

Proof

The identity contributes $t_1^{|G|/|H|}$ to the sum of the monomials. All the (q^2-1) parabolics lie in the same conjugacy class, hence they have the same monomials. So from Table 3.1.2 the contributions by elements of τ_1 is $(q^2-1)t_p^{p^{f-1}(q+1)}$. Each $g \in \tau_2$ is contained in a unique cyclic subgroup C_{q-1} and there are in total $\frac{q(q+1)}{2}$ conjugates of C_{q-1}. Hence by Theorem 4.1.1 and the results in Table 3.1.2 the contributions by elements of τ_2 is $\frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_1^2 t_d^{\frac{(q-1)(q+2)}{d}}$. Finally each $g \in \tau_0$ is contained in unique cyclic subgroup C_{q+1} and there are in total $\frac{q(q-1)}{2}$ conjugates of C_{q+1} so they contribute

$\frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_d^{\frac{q(q+1)}{d}}$ to the same of the monomial. Adding all the contributions and dividing by the order of G we get the desired results.■

3.2 Cycle index of G acting on the cosets of $H = C_{q+1}$

Theorem

The cycle index of $PGL(2,q)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{|G|/|H|} + (q^2-1)t_p^{p^{f-1}(q-1)} + \frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_d^{\frac{q(q-1)}{d}} + \frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_1^2 t_d^{\frac{(q+1)(q-2)}{d}}\right].$$

Proof

Using the results in Table 2.2.2 and argument similar to those in Theorem 3.1 the result is immediate. ■

3.3 Cycle index of G acting on the cosets of $H = P_q$

From the results in Table 2.3.2, we have the following theorem;

Theorem

The cycle index of G on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{|G|/|H|} + (q^2-1)t_1^{q-1}t_p^{p^{f-1}(q-1)} + \frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_d^{\frac{1}{d}(q^2-1)} + \frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_d^{\frac{1}{d}(q^2-1)}\right]$$

REFERENCES

Dickson, L. E. (1901). *Linear groups with an exposition of the Galois field theory*. Teubner, Dover, Newyork

Hardy G. H. and Wright E. M. (1938). *An introduction to the theory of numbers*. Oxford University Press.

Huppert B. (1967). *Endlichegruppen I*. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer, Berlin.

Kamuti, I. N. (1992). *Combinatorial formulas, invariants and structures associated with primitive permutation representations of PSL (2, q) and PGL (2, q)*, Ph. D. Thesis, Southampton University, U.K.

Redfied, J. H. (1927). The theory of group- reduced distributions. *Journal of American Mathematics Soc*iety. 49 : 433-455.