Comparison of Godunov's and Relaxation Schemes Approximation of Solutions to the Euler Equations

S.K. Mutua[1] and M.E. Kimathi[2]

Abstract

In this paper we deal with the study of Euler equations for isothermal gas that is governed by two hyperbolic equations. By analysing the equations we obtain two real and distinct eigenvalues which enables us to determine the wave structure of the possible solution to the Riemann problem set up. We then obtain the numerical solution to the Riemann problem that we set up using the Godunov scheme and the relaxation scheme. Finally, we compare the results obtained from these two schemes graphically and explain in details.

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[1] Pure & Applied Mathematics Department, Jomo Kenyatta University of Agriculture & Technology, Nairobi, Kenya. E-mail: mutuasamuel@gmail.com
[2] Pure & Applied Mathematics Department, Technical University of Kenya, Nairobi, Kenya. E-mail: memkimathi@gmail.com

1 Introduction

Isothermal gas is one in which the temperature throughout the gas remains constant and thus the ideal gas law is reduced to $p = a^2\rho$. That is pressure is a function of density alone. Consider the Euler equation for isothermal gas. The system consists of Euler equations and is strictly hyperbolic with two real and distinct eigenvalues, whereby, one is greater than the other. Eigenvalues physically represent speeds of propagation of information. Depending on the initial data the eigenvalues may represent shock and rarefaction waves. Riemann problem consist of equations together with the discontinuous initial data consisting of two constant states separated by a single discontinuity. It is very useful in understanding hyperbolic partial differential equations because all properties appear as characteristics in the solution. It is defined by:

$$u(x, t = 0) = \begin{cases} u_L & x \leq 0 \\ u_R & x > 0 \end{cases}$$

Using local relaxation approximation Shi Jin *et al*[5] constructed a linear hyperbolic system with a stiff lower order term that approximates the original system with a small dissipative correction. The main feature of this class of schemes was its simplicity and generality since it used neither Riemann solvers spatially nor non-linear systems of algebraic equations solvers temporally, yet it could achieve high order accuracy and picked up the right weak solutions. Also Godunov proposed a way to make use of the characteristic information within the framework of a conservative method by suggesting solving the Riemann problems forward in time and the solutions were easy to compute as well as gave substantial information about the characteristic structure and lead to conservation methods since they were themselves exact solutions of the conservation laws and hence conservative, Leveque, [4].

2 Mathematical formulation of the Euler equations

Considering the one-dimensional time dependent Euler equation for the conservation laws given by

$$U_t + F(U)_x = 0 \tag{2.1}$$

where $\qquad U = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} \qquad\qquad F = \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix}$

with initial conditions

$$U(x,0) = \begin{cases} U_L & x \le 0 \\ U_R & x > 0 \end{cases} \tag{2.2}$$

In solving the Riemann problem, we shall frequently make use of the vector $W = (\rho, u)$ of primitive variables rather than the vector U of conserved variables, where is ρ density and u is the particle velocity, Toro, [1].

Data consists of just two constant states which in terms of primitive variables are $W_L = (\rho_L, u_L)$ to the left of $x = 0$ and $W_R = (\rho_R, u_R)$ to the right of $x = 0$ and separated by a discontinuity at $x = 0$. For the case in which no vacuum is present the exact solution of the Riemann problem has two waves which are associated with the eigenvalues $\lambda_1 = u - a$ and $\lambda_2 = u + a$, Mutua S.K *et al,* [2].

From the above eigenvalues and their corresponding eigenvectors then we can be able to determine the structure of the waves since the two waves separate three constant states namely W_L (data on the left hand side), W_* and W_R (data on the right hand side). The solution to this problem depends on the relative values of W_L and W_R. That is, for $W_L < W_R$, a rarefaction wave is going to develop while for $W_L > W_R$, a shock wave is going to appear. All the two quantities, that is, density and particle velocity change across a shock wave. Considering a right facing shock wave travelling at the constant speed, S. In terms of the primitive variables we denote the state ahead of the shock by $W_R = (\rho_R, u_R)^T$ and the state behind the shock by $W_* = (\rho_*, u_*)^T$. Now we are interested in deriving relations across the

shock wave between the various quantities involved. We accomplish this by finding the intermediate state such that they are connected by a discontinuity satisfying the Rankine-Hugoniot condition, given by:

$$\rho_l u_l - \rho_* u_* = S(\rho_l - \rho_*)$$
$$(\rho_l u_l^2 - a^2 \rho_l) - (\rho_* u_*^2 - a^2 \rho_*) = S(\rho_l u_l - \rho_* u_*)$$

Now introducing a mass flux, Q, and using the above equations to do some manipulation one can be able to obtain the shock speed S, Leveque [4], given as:

$$S = u_* \pm a \sqrt{\frac{\rho_l}{\rho_*}} \tag{2.3}$$

Similarly, inspection of the eigenvectors for the primitive variable formulation reveals that density and particle velocity change across a rarefaction wave. Now utilizing the i-th generalized Riemann invariants given by the (M-1) ODEs, we have

$$U = (\rho, \rho u)^T, \quad \lambda_1 = u - a, \quad \lambda_2 = u + a$$

$$K^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix}, \quad K^{(2)} = \begin{bmatrix} 1 \\ u + a \end{bmatrix}$$

$$\therefore \quad \frac{du_1}{K_1^{(i)}} = \frac{du_2}{K_2^{(i)}} \tag{2.4}$$

$$\Rightarrow \frac{d\rho}{1} = \frac{d(\rho u)}{u - a} \Rightarrow du + \frac{a}{\rho} d\rho = 0 \Rightarrow u + \int \frac{a}{\rho} d\rho = 0 \Rightarrow u + a \ln \rho = C$$

$$\Rightarrow u - a \ln \rho = C$$

where C is a constant.

Now using the above equations and some manipulations we obtain ρ_* and u_* as shown below

$$u_* + a \ln \rho_* = u_L + a \ln \rho_L$$
$$u_* - a \ln \rho_* = u_R - a \ln \rho_R$$

$$\therefore 2u_* = u_L + u_R + a\,ln\,\rho_L - a\,ln\,\rho_R$$

$$2u_* = u_L + u_R + a\left(ln\,\rho_L - ln\,\rho_R\right) \Rightarrow u_* = \frac{u_L + u_R + a\left(ln\,\rho_L - ln\,\rho_R\right)}{2} \quad (2.5)$$

Similarly,

$$2a\ln\rho_* = u_L - u_R + a\ln\rho_L + a\ln\rho_R$$

$$\ln\rho_* = \frac{u_L - u_R + a\left(\ln\rho_L + \ln\rho_R\right)}{2a} \quad (2.6)$$

$$\therefore \rho_* = e^{\frac{u_L - u_R + a\left(\ln\rho_L + \ln\rho_R\right)}{2a}}$$

Suppose we wish to solve the Riemann problem with left and right states u_L and u_R. Just as in the linear case, we can accomplish this by finding an intermediate state u_* such that u_L and u_* are connected by a discontinuity satisfying the Rankine-Hugoniot condition and so are u_* and u_R Leveque, [4].

3 Simulation

3.1 Relaxation Scheme

The details of the numerical scheme can be found at Mutua S.K *et al* [3] We use the value of h as 0.005 but choose the time step, k, according to Courant-Friedrichs-Lewy (CFL) condition.

$$\frac{k}{h|\lambda_{\max}|} \le \frac{1}{2}$$

where $\lambda_{\max}$ is the maximal (in absolute value) eigenvalue of the Jacobian matrix.

Now investigating two Euler flow scenarios that lead to two solutions of interest namely 1-Rarefaction followed by 2-Shock wave and a 1-Shock wave followed by 2-Rarefaction.

In the first scenario we consider the initial data $\rho_L = 0.9, \rho_R = 0.2$ $v_L = 0.1, v_R = 0.2$. Figure 3.10 (a), 3.10(b), 3.10(c) shows the density, Space – time plot and velocity profiles respectively
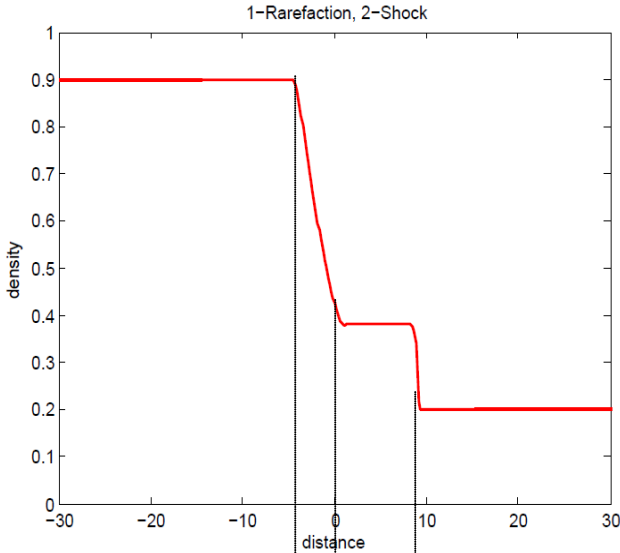

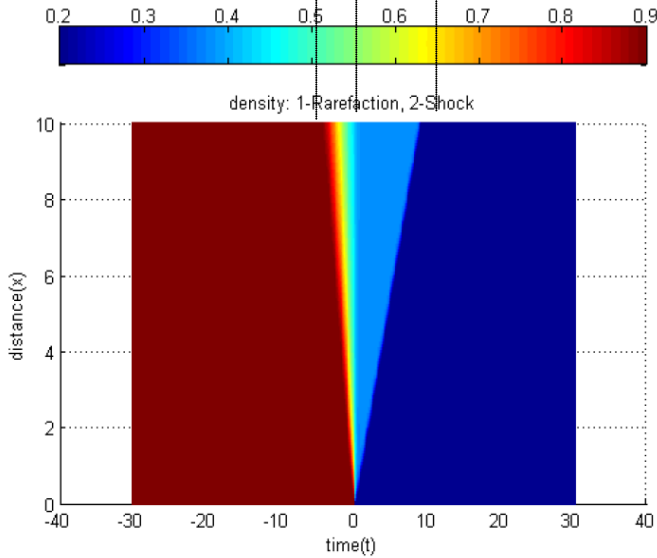Figure 3.10 (a): Density profile for a 1-rarefaction followed by a 2-shock

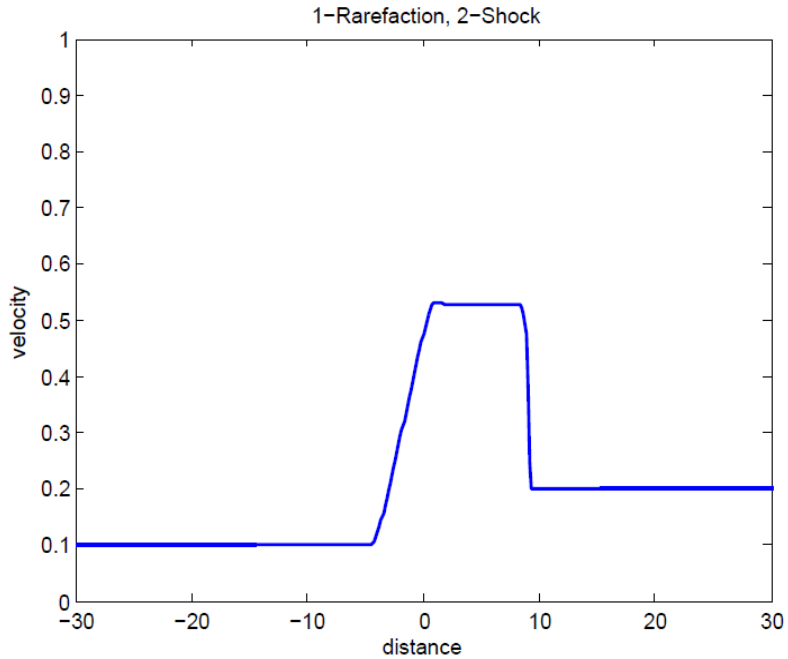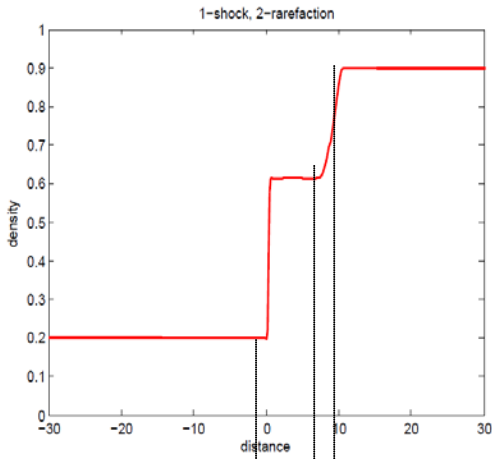Figure 3.10 (b): distance-time graph of the density profile for a 1-rarefaction followed by a 2-shock

Figure 3.10 (c): Velocity profile for a 1-rarefaction followed by a 2-shock

In the second scenario we consider the initial data $\rho_L = 0.2, \rho_R = 0.9$, $v_L = 0.9, v_R = 0.5$. Figure 3.11 (a), 3.11(b), 3.11(c) shows the density, Space – time plot and velocity profiles respectively.

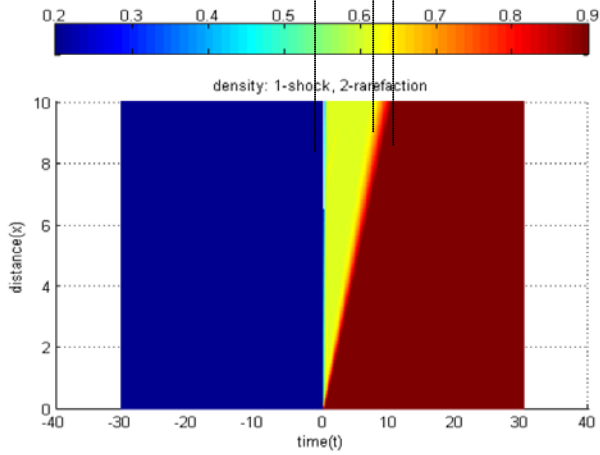Figure 3.11 (a): Density profile for a 1-shock followed by a 2-rarefaction.

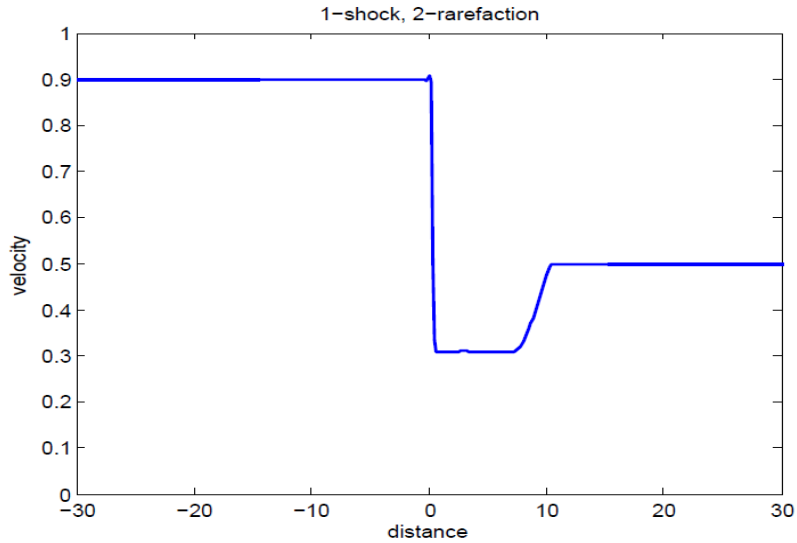Figure 3.11 (b): distance-time graph of the density profile for a 1-shock followed

by a 2-rarefaction

Figure 3.11 (c): Velocity profile for a 1-shock followed by a 2-rarefaction.

3.2 Godunov Scheme

The details of the numerical scheme can be found at Mutua S.K *et al*, [3]. Also using the value of h as 0.005 but choose the time step, k, according to Courant-Friedrichs-Lewy (CFL) condition, we investigate two Euler flow scenarios that lead to two solutions of interest using the Godunov scheme. In the first scenario we consider the initial data $\rho_L = 0.9, \rho_R = 0.2 \; v_L = 0.1, v_R = 0.2$. Figure 3.12(a) and 3.12(b) shows the density and velocity profiles for the wave solutions respectively.
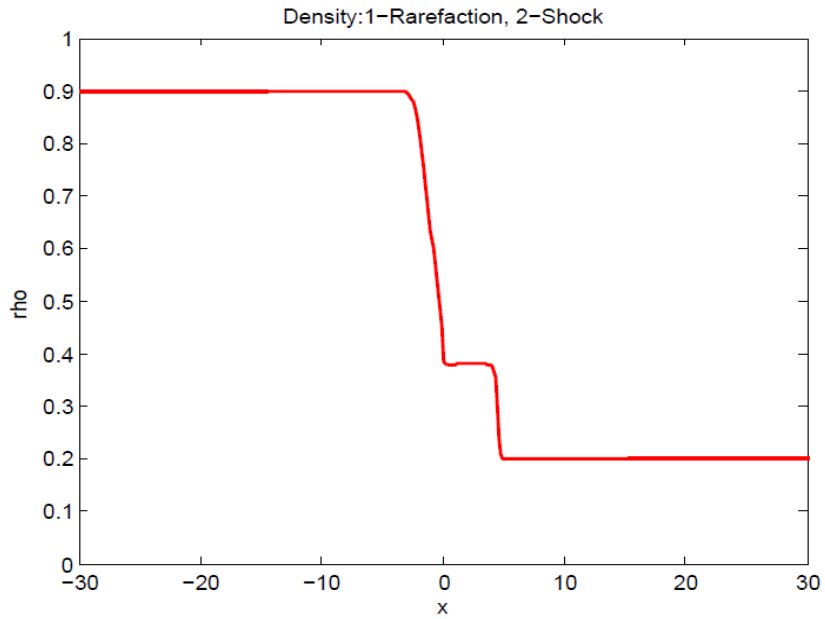
Figure 3.12 (a): Density profile for a 1-rarefaction followed by a 2-shock
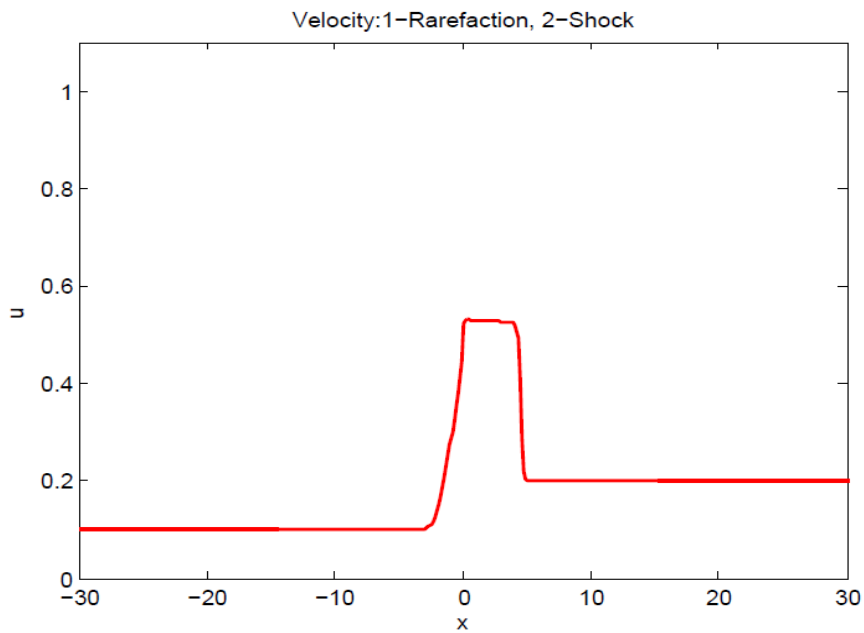

Figure 3.12 (b): Velocity profile for a 1-rarefaction followed by a 2-shock

In the second scenario we consider the initial data $\rho_L = 0.2, \rho_R = 0.9$, $v_L = 0.9, v_R = 0.5$ to yield a 1-Shock followed by 2-Rarefaction wave as shown below for both density and velocity profiles respectively.
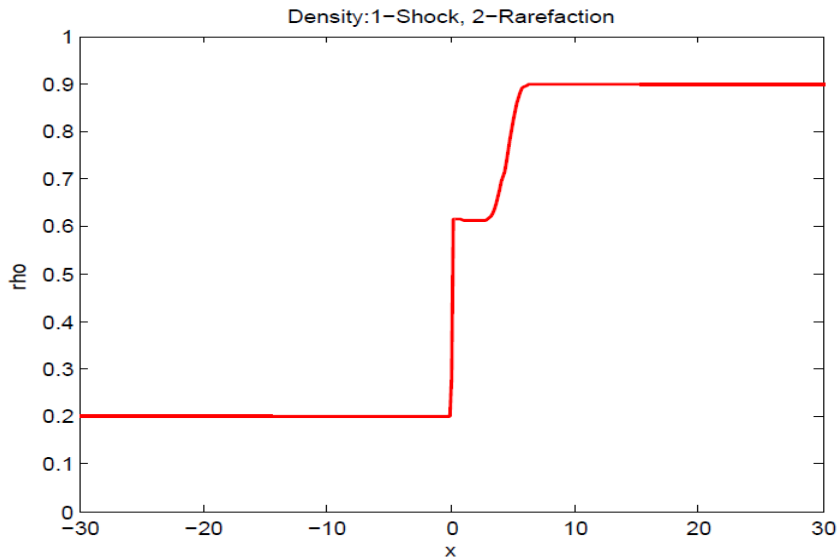

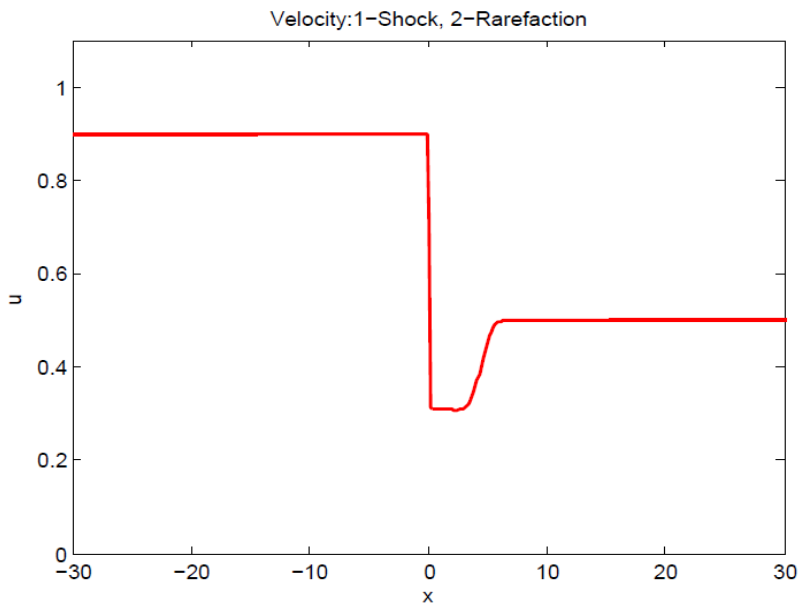Figure 3.13 (a): Density profile for a 1-shock followed by a 2-rarefaction

Figure 3.13 (b): Velocity profile for a 1-shock followed by a 2-rarefaction

4 Comparison

Having considered both the density and velocity profiles for the Euler equations using both the Relaxation and Godunov's schemes, we now present a comparison for the two schemes as shown in the figures below.

Figure 3.14(a) shows the density profile comparison for both schemes applied to Euler equations for a 1-Rarefaction wave followed by 2- shock wave solution. Figure 3.14(b) shows the velocity profiles for the same wave solutions.
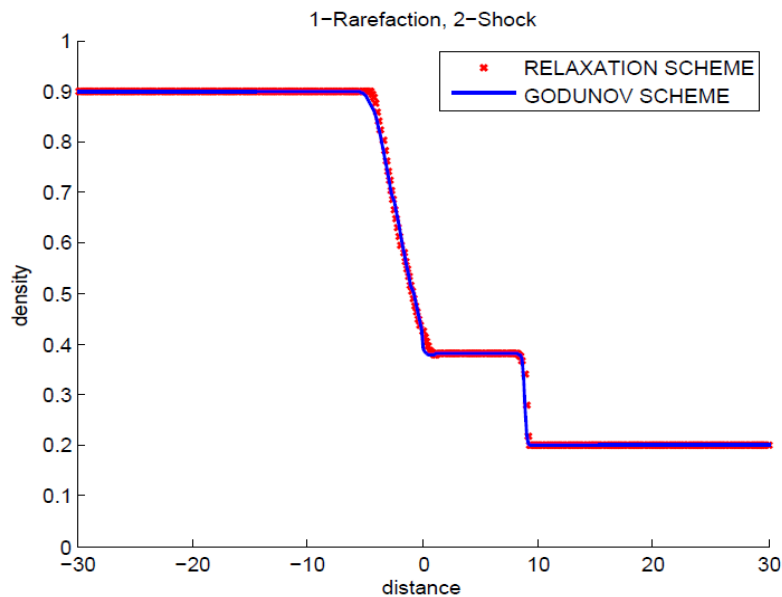

Figure 3.14 (a): Density profile comparison for 1-Rarefaction followed by 2-Shock
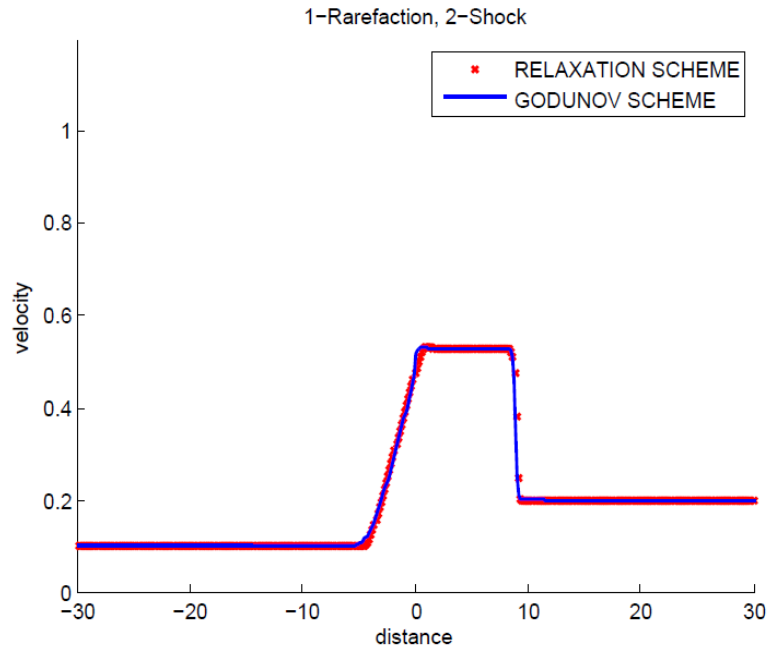
Figure 3.14 (b): Velocity profile comparison for 1-Rarefaction followed by 2-Shock

Figure 3.15(a) shows the density profile comparison for both schemes applied to Euler equations for a 1-Shock wave followed by 2- Rarefaction wave solution. Figure 3.15(b) shows the velocity profiles for the same wave solutions.
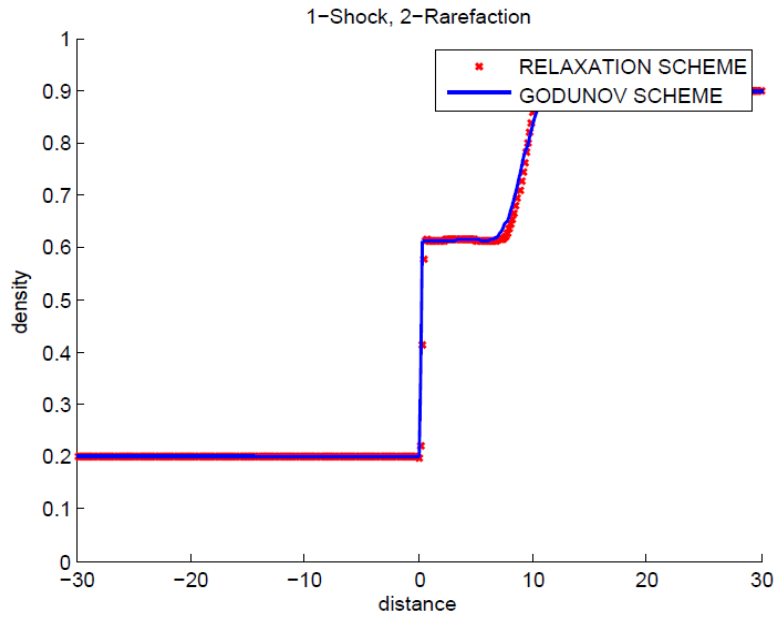
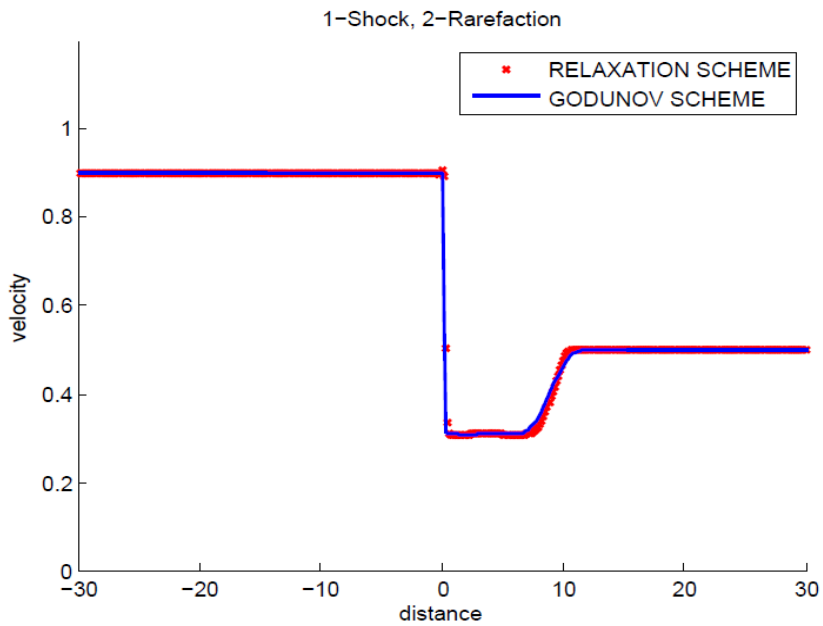Figure 3.15 (a): Density profile comparison for 1-Shock followed by 2-Rarefaction

Figure 3.15 (b): Velocity profile comparison for 1-Shock followed by 2-Rarefaction

5 Conclusion

In this study Euler equations have been studied and simulated using both Godunov and relaxation schemes and their respective results compared using graphs.

From the graphs above it is noted that the relaxation scheme performs equally better as the Godunov scheme. Thus it appears to be more promising and a good alternative to the Godunov scheme because of its simplicity.

Due to insufficient time the authors of this paper decided to consider both numerical schemes that is Godunov scheme and relaxation schemes and thus the authors recommends for comparison of the schemes with the exact solution to check their accuracy so as to show which scheme is more accurate than the other in terms of approximating the exact solutions.

References

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[5] Shi Jin, Zhouping Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, John Wiley & Sons, Inc., 235-275, 1995.