MACHAKOS UNIVERSITY

University Examinations for 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECOND YEAR FIRST SEMESTER EXAMINATIONS FOR

BACHELOR OF SCIENCE (MATHEMATICS AND STATISTICS)

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF EDUCATION

SMA 260: PROBABILITY AND STATISTICS I

DATE: **TIME:**

INSTRUCTIONS TO CANDIDATES.

Attempt Question ONE and any other TWO questions.

QUESTION ONE (COMPULSORY)(30 MARKS)

a) Let X have the pdf, $f(x) = \begin{cases} \frac{1}{2}(x+1) & -1 < x < 1 \\ 0 & elsewhere \end{cases}$

i. Obtain the expectation of X. (2 marks)

ii. Obtain the variance of X. (2 marks)

b) Let x be $N(\mu, \sigma^2)$. Compute $p(\mu - 2\sigma < x < \mu + 2\sigma)$. (3 marks)

c) By definition, $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$. Use this definition to show that, $\Gamma(\alpha = 1) = \Gamma(\alpha = 2)$ (2 marks)

d) An architect claims that only 40% of the multi-storey buildings in residential areas in Machakos were approved by a registered engineer. Assuming that this claim is true, determine the probability that among 12 such buildings randomly selected from the residential areas, the following were approved by a registered engineer:

i. Exactly 4 building. (2 marks)

ii. Between 4 and 6 inclusive. (4 marks)

iii. At least 3 buildings. (4 marks)

e) If a random variable x has a pdf given by

$$f(x) = \begin{cases} 0.81^x(1-0.81)^{1-x} & ; \ x = 0,1 \\ 0 & ; \ elsewhere \end{cases}$$

i. What is the distribution called? (1 mark)

ii. Identify the parameter in this distribution. (1 mark)

iii. Obtain the mean and the variance. (2 marks)

f) Two fair six faced dice are tossed. A random variable x is defined as the difference between the outcomes of the two dice.

i. Derive the sample space for the random variable x. (2 marks)

ii. Derive the probability distribution function of x. (2 marks)

iii. Determine the mean and the variance of x. (3 marks)

QUESTION TWO (20 MARKS)

a) Given the p.d.f of a random variable x,

$$f(x) = \begin{cases} me^{-x} & ; \ x > 0 \\ 0 & ; otherwise \end{cases}$$

i. Determine the value of m for this function to serve as a p.d.f. (1 mark)

ii. Determine the moment generating function (mgf) of x? (2 marks)

iii. Use the mgf obtained in (ii) to compute the mean and the variance of x. (4 marks)

b) Suppose x is a discrete random variable with a binomial probability distribution.

I. Obtain the mgf of x (2 marks)

II. Using the idea in (i), if the mgf of a random variable x is $M(t) = \left[\frac{2}{3} + \frac{1}{3}e^t\right]^5$;

i. obtain the pmf of this distribution. (2 marks)

ii. Calculate the mean and variance. (2 marks)

c) Given the mgf of a random variable x as $e^{2(e^t - 1)}$;

i. Prove that the x has mean equal to the variance. (4 marks)

ii. Write down the p.m.f of x and prove that it satisfies the conditions necessary for a p.m.f. (3 marks)

QUESTION THREE (20 MARKS)

a) It is known that 4% of the items coming out of a production process are defective. Determine the probability that among 250 items randomly selected from the production process:

i. Exactly 5 are defective. (2 marks)

ii. At least 5 are defective. (6 marks)

iii. Between 2 and 4 inclusive are defective. (2 marks)

b) There is an outbreak of a disease whose diagnosis is not perfectly accurate. A random sample of 140 patients was taken out of whom 32 patients had the disease. A total of 34 patients were subjected to a laboratory diagnostic test.

I. Determine the probability for each of the following:

i. That 10 of the patients who were tested actually had the disease. (2 marks)

ii. That between 6 and 9 inclusive of the patients who were tested actually had the disease. (5 marks)

II. A random variable x is defined as the number of patients who were tested and actually had the disease. Determine the mean and the variance of the random variable x. (3 marks)

QUESTION FOUR (20 MARKS)

a) The marks scored in SMA 260 by students who sat for the examination in a certain year has been found to be normally distributed with mean 54 marks and a standard deviation (12 marks)

i. Determine the proportion of the students who passed if the pass-mark was set at 45 marks. (3 marks)

ii. Determine the proportion of the students who scored a credit grade if a credit is assigned for marks between 60 and 74. (2 marks)

iii. If the top 65% of the students are supposed to pass the examination, determine the mark which should be set as the pass-mark to achieve this. (2 marks)

iv. Grades for results are awarded based on the following criteria:

- Fail to the bottom 20%
- Pass to the next 35%
- Credit to the next 30%
- Distinction to the top 15%

Determine the lower and upper limits of the range of the marks for any two criteria's above. (4 marks)

b) Given that x is a random variable with p.d.f

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} & ; \quad 0 < x < \infty \\ 0 & ; \ elsewhere \end{cases}$$

i. Identify this distribution? (1 mark)

ii. Convert this density function to that of a random variable x that follows the chi-square distribution. (2 marks)

iii. Given that the mgf of x in (i) is $\frac{1}{(1-\beta t)^{\alpha}} = (1-\beta t)^{-\alpha}$ for $t < \frac{1}{\beta}$, what is the mgf of x in (ii)? (2 marks)

iv. If the mean and variance of x in (i) are given by 6 and 12, what is α in (i) and the mean and variance of x in (ii)? (4 marks)

QUESTION FIVE (20 MARKS)

a) Differentiate between discrete and continuous random variables. (2 marks)

b) Given a continuous random variable x with p.d.f. given by $f(x) = \begin{cases} 2x & ; \quad 0 < x < 1 \\ 0 & ; \quad elsewhere \end{cases}$

Let Y be another continuous random variable defined by $Y = 8x^3$. Determine the following about the distribution of y.

i. The probability density function of y. (4 marks)

ii. The mean and the variance of y. (5 marks)

iii. The probability $p(y > 4)$. (2 marks)

c) If a random variable x has a p.d.f,

$$f(x) = \begin{cases} 1/2, & -1 < x < 1 \\ 0, & elsewhere \end{cases}$$

i. Find the cumulative distribution function (c.d.f) of the random variable x. (2 marks)

ii. Find the $p(-0.5 < x < 0.3)$. (2 marks)

d) Given a random variable x and c.d.f. given by

$$F(x) = \begin{cases} 0 & ; \quad x < 0 \\ \frac{2}{5}x + 4 & ; \quad 0 \leq x \leq 5 \\ 1 & ; \quad x > 5 \end{cases}$$

Determine the p.d.f of x. (3 marks)