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"On Irreducible Representations of the Symmetric Group"

A PROJECT REPORT SUBMITTED TO THE SCHOOL OF MATHEMATICS IN PARTIAL FULFILLMENT FOR A MASTER OF SCIENCE DEGREE IN PURE MATHEMATICS

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I the undersigned declare that this project report is my original work and to the best of my knowledge has not been presented for the award of a degree in any other university

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"Dedicated to my mother - whose love kept me going"

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Abstract

In chapter 1 and 2, We begin by introducing a few elementary aspects of the symmetric group S_n and there after we give a brief discussion on the representation theory with emphasis on the symmetric group. In chapter 3. we discuss the construction of an ordinary irreducible S_n-submodule basing on the format provided in the book by Sagan [6]. Finally. we discuss on the irreducibility of the constructed submodule as the base field shifts from a field of characteristic 0 to a field of characteristic $p > 0$

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Introduction

As mentioned in [8], A group representation can be thought of as an action of a group G on some vector space. Such actions arise naturally in many branches of mathematics and physics and as such it is important to study and understand the theory of representations. Cayley's theorem asserts that every finite group can be embedded in a group of symmetries S_n for some n - this makes the studies related to the symmetric group of much significance since by the isomorphism guaranteed by the Cayley's theorem we may understand the properties of any other finite group.

As an illustration, the solution space of a differential equation in a 3-dimensional space having a rotational symmetry has its solution space invariant under rotations. Thus the space of solutions will constitute a representation of the rotation group $SO(3)$. If we know what all of the representations of $SO(3)$ are, then this can help immensely in narrowing down what the space of solutions can be.

In fact, one of the chief applications of representation theory is to exploit symmetry. If a system has symmetry, then the set of symmetries will form a group, and understanding the representations of the symmetry group allows us to use that symmetry to simplify the problem

This report focuses on the representation theory of symmetric groups and in particular on the construction of all irreducible modules of the symmetric group S_n (*otherwise known as the Specht Modules*). There are numerous applications of representation of groups and in particular the representation of the symmetric group. For instance, they arise in physics, probability and statistics, topological graph theory, the theory of partially ordered sets amongst many other areas.

This report contains 5 chapters - in chapter 1 we have provided elementary definitions as well as basic results about the symmetric group S_n. Chapter 2 is on the ordinary representation of finite group with some small emphasis on the symmetric group S_n. It should be noted that this chapter is not exhaustive as far as representation theory is concerned. we have only mentioned some useful aspects that is in direct reference to the main objective of this report. In chapter 3 we have dwelt on the construction of the ordinary irreducible modules of the symmetric group S_n popularly known as the Specht modules. and a simple example for the case $n = 3$ has been given.

In representation theory of finite groups, it is useful to know which ordinary irreducible representations remain irreducible when reduced modulo a prime p. In chapter 4. we have traced the history of classification of ordinary irreducible modules that remain irreducible modulo p.

Chapter 1

The Symmetric Group S_n

1.1 Introduction

Definition 1.1.1 (The Symmetric Group) Let $S = \{a_1, a_2, \ldots a_n\}$ be a set of n distinct elements. For Convenience we label the elements as $\{1, 2, \ldots, n\}$. The set of all permutations of elements of S forms a group under the binary operation of composition. This group is denoted by S_n and is called the *symmetric group of degree n*.

Herstein [1] and Fraleigh [2] have a fairly elementary treatment of symmetric groups.

1.2 Partitions

Permutations may be given using different types of notations.

Definition 1.2.1 A cycle is a permutation of the following form

$$i_1 \mapsto i_2 \mapsto i_3 \mapsto \cdots \mapsto i_k \mapsto i_1$$

where i_j are all distinct. This cycle is denoted by $(i_1, i_2, i_3, \ldots i_k)$ and is called a *k-cycle* or a cycle of length k.

A cycle of length 2 is called a *transposition*. It can be shown easily that a permutation can be written as a product of disjoint cycles; on the other hand, every cycle is a product of transpositions. Consequently, every permutation can be written as a product of transpositions.

Definition 1.2.2 The *signature* of a permutation π denoted $\mathrm{sgn}(\pi)$ is defined to be

$$\mathrm{sgn}(\pi) = (-1)^r$$

where r is the number of transpositions in the decomposition of π in terms of transpositions.

An example of a cycle notation of a permutation is

$$\pi = (123)(45)(6)(7) \tag{1.2.1}$$

The cycle type of a permutation π is an expression of the form

$$(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$$

where m_k is the number of cycles of length k in π. The permutation in the example given in 1.2.1 above has the type

$$(1^2, 2^1, 3^1, 4^0, 5^0, 6^0, 7^0)$$

We can also use a *partition* to give the cycle type. In 1.2.1 above, π is a product of four cycles: a 3-cycle, a 2-cycle and two 1-cycles. This information can be represented as $(3, 2, 1, 1)$ or more conveniently using exponential notation as $(3, 2, 1^2)$.

Definition 1.2.3 (Partition) A partition λ of a positive integer n is a sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

such that $\lambda_i \geq \lambda_{i+1}$ for all integers $1 \leq i \leq l$ and $\sum_{i=1}^{l} \lambda_i = n$. The entry λ_i is called a *part* of λ

Definition 1.2.4 (Conjugate Partition) For each partition λ of n there is a conjugate partition of n, denoted $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_k)$ where

$$\lambda'_i = \max\{j | \lambda_j \geq i\}$$

Example 1.2.1 The permutation given in 1.2.1 has a partition of the type $\lambda = (3, 2, 1^2)$. Its conjugate partition is then $\lambda' = (4, 2, 1)$

1.3 Conjugacy Classes of S_n

Definition 1.3.1 Let $g, h \in G$. We say that *g is conjugate to g in G* if

$$h = xgx^{-1} \quad \text{for some } x \in G$$

The set g^G of all elements conjugate to h in G is called the *conjugacy class* of g in G.

Distinct conjugacy classes are disjoint and therefore a group G can be written as

$$G = g_1^G \cup g_2^G \cup \cdots \cup g_s^G$$

that is as a union of distinct conjugacy classes.

Definition 1.3.2 If H is a subgroup of G we say that $g_1, g_2, \ldots, g_k$ is a *transversal* for H if $\mathcal{H} = \{g_1H, g_2H, \ldots, g_kH\}$ is a complete set of disjoint left cosets for H in G

Considering the case where $G = S_n$, we have the following simple observation:

Proposition 1.3.1 Let x be a k-cycle $(i_1, i_2, \ldots i_k)$ in S_n and let $g \in S_n$ then gxg^{-1} is the k-cycle $(gi_1, gi_2, \ldots, gi_k)$

Proof: Write $A = \{i_1, \ldots, i_k\}$. For $i_r \in A$.

$$(gxg^{-1})gi_r = gxi_r = gi_{r+1} \text{ (or } gi_1 \text{ if } r = k) \tag{1.3.1}$$

also, for $1 \leq i \leq n$ and $i \notin A$

$$(gxg^{-1})gi = gxi = gi \tag{1.3.2}$$

Hence $g(i_1, \ldots, i_k)g^{-1} = (gi_1. gi_2, \ldots, gi_k)$ as required.

Now considering an arbitrary permutation $x \in S_n$, write $x = (a_1, \ldots a_{k_1})(b_1, \ldots b_{k_2}) \ldots (c_1, \ldots c_{k_s})$ as a product of disjoint cycles with $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_s$. By the above proposition, for $g \in S_n$ we have

$$gxg^{-1} = (ga_1, \ldots ga_{k_1})(gb_1, \ldots gb_{k_2}) \ldots (gc_1, \ldots gc_{k_s})$$

The s-tuple $\lambda = (k_1, k_2, \ldots, k_s)$ is called the *cycle-shape* of x. It is also called a partition of n. We note that x and gxg^{-1} have the same cycle-shape. On the other hand given any two permutations x, y having the same cycle-shape, say

$$x = (a_1, \ldots a_{k_1})(b_1, \ldots b_{k_2}) \ldots (c_1, \ldots c_{k_s})$$
$$y = (\acute{a}_1, \ldots \acute{a}_{k_1})(\acute{b}_1, \ldots \acute{b}_{k_2}) \ldots (\acute{c}_1, \ldots \acute{c}_{k_s})$$

then there exist $g \in S_n$ sending $a_1 \rightarrow \acute{a}_1, \ldots, c_{k_s} \rightarrow \acute{c}_{k_s}$ hence

$$gxg^{-1} = y.$$

Thus for $x \in S_n$, the conjugacy class x^{S_n} of x in S_n consists of all permutations in S_n which have the same cycle-shape as x. From this we see that the number of possible partitions of n is equal to the number of distinct conjugacy classes in S_n. This fact is used to show that the exact number of non-isomorphic irreducible representations of S_n is equal to the number of distinct conjugacy classes of S_n

Chapter 2

Representations and Characters of Finite Groups

2.1 Representations of a Group G

Throughout this report, it will be assumed that G is a finite group and V is a finite dimensional vector space over a field $\mathbb{F}$.

Representation theory can be approached from two different angles - either in terms of matrices or or in terms of modules.

Definition 2.1.1 A *representation* of a group G is a group homomorphism T from G to the group $GL(V)$ of all invertible linear transformations of V, hence we say V a G-module.

Definition 2.1.2 (Matrix Representation) A *matrix representation* of a group G is a group homomorphism X from G to the group $GL(n, \mathbb{F})$ of all invertible $n \times n$ matrices.

Definition 2.1.3 If dim $V = n$ we say that the representation T has *degree* or *dimension* n, similarly if X is a matrix representation of G mapping elements of G to the group of $n \times n$ invertible matrices over $\mathbb{F}$ then we say that X is of dimension n.

Definition 2.1.4 Alternatively, we say that V is a G-module if there exists a multiplication gv between the elements $v \in V$ and $g \in G$ satisfying these properties:

1. $gv \in V$,
2. $g(c_1 v + c_2 w) = c_1(gv) + c_2(gw)$, where $c_i \in \mathbb{F}$, $w \in V$
3. $(gh)v = g(hv)$, $\quad h \in G$
4. $\epsilon v = v$

Consequently we get a correspondence between a matrix representation X of degree n and the vector space $\mathbb{F}^n$, we simply define the multiplication gv by

$$gv \stackrel{def}{=} X(g)v \tag{2.1.1}$$

It can be verified easily that the multiplication defined satisfies the above properties in 2.1.4 and therefore show that $\mathbb{F}^n$ is a G-module. On the other hand, if V is a G-module and $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ a basis for V, then the mapping $v \to gv$ is an endomorphism. If we define $X(g) = [g]_{\mathcal{B}}$ where $[g]_{\mathcal{B}}$ is the matrix of endomorphism relative to the basis $\mathcal{B}$, then the mapping defined is a matrix representation of G.

Example 2.1.1 Let G be an arbitrary group. Let X be a mapping that assigns to each element $g \in G$ the 1×1 identity matrix (1). This representation of dimension 1 and is called the trivial representation and denoted by 1_G or just 1 when there is no ambiguity about the group G in context. When

$G = S_n$ another important representation of dimension 1 is the signature representation given by

$$X(\pi) = \text{sgn}(\pi) \text{ for all } \pi \in S_n$$

This representation is usually denoted by **sgn**

Example 2.1.2 Let G act on itself by left multiplication and let $\mathbb{F}G = \mathbb{F}\{g_1, g_2, \ldots, g_n\}$ denote the group algebra generated by G over $\mathbb{F}$; that is $\mathbb{F}G$ consist of all formal linear combinations

$$c_1\mathbf{g}_1 + c_2\mathbf{g}_n + \cdots + c_n\mathbf{g}_n,$$

where $c_i \in \mathbb{F}$ for all i (The elements of G are put in boldface to show that they are being considered as vectors). Vector addition and scalar multiplication in $\mathbb{F}G$ are defined by

$$(c_1\mathbf{g}_1 + \cdots + c_n\mathbf{g}_n) + (d_1\mathbf{g}_1 + \cdots + d_n\mathbf{g}_n) = (c_1 + d_1)\mathbf{g}_1 + \cdots + (c_n + d_n)\mathbf{g}_n$$

and

$$c(c_1\mathbf{g}_1 + \cdots + c_n\mathbf{g}_n) = (cc_1)\mathbf{g}_1 + \cdots + (cc_n)\mathbf{g}_n$$

The action of G on itself can be extended to an action on $\mathbb{F}G$ by linearity:

$$g(c_1\mathbf{g}_1 + \cdots + c_n\mathbf{g}_n) = c_1(g\mathbf{g}_1) + \cdots + c_n(g\mathbf{g}_n)$$

for all $g \in G$. This makes $\mathbb{F}G$ a G-module of dimension $|G|$.

To make things more concrete we consider the dihedral group $D_4 = \langle a, b : a^2 = b^4 = 1, ba = ab^3 \rangle$ where 1 denote the identity element and let $\mathbb{F} = \mathbb{C}$

$$\mathbb{C}D_4 = \{c_1\mathbf{1} + c_2\mathbf{b} + c_3\mathbf{b}^2 + c_4\mathbf{b}^3 + c_5\mathbf{a} + c_6\mathbf{ab} + c_7\mathbf{ab}^2 + c_8\mathbf{ab}^3 : c_i \in \mathbb{C} \text{ for all } i\}$$

To find the matrix of ab^3 with respect to the standard basis $\mathcal{B} = G$ we find the matrix of endomorphism when ab^3 acts on the basis elements $\mathcal{B} = \{1, \mathrm{b}, \mathrm{b}^2, \mathrm{b}^3, \mathrm{a}, \mathrm{ab}, \mathrm{ab}^2, \mathrm{ab}^3\}$.

$ab^3 1 = ab^3, ab^3\mathrm{b} = a, ab^3\mathrm{b}^2 = ab, ab^3\mathrm{b}^3 = ab^2, ab^3\mathrm{a} = b, ab^3\mathrm{ab} = b^2, ab^3\mathrm{ab}^2 = b^3, ab^3\mathrm{ab}^3 =$

Thus

$$X(ab^3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

After computing the matrices corresponding to the remaining elements of D_4 we find that all the matrices are permutation matrices and are all distinct. The mapping X mapping elements of D_4 to the group of $GL(8, \mathbb{C})$ is a matrix representation of D_4. This representation is known as a *regular representation* and has a lot of significance in representation theory. Furthermore, a matrix representation of a group G mapping all the elements of G to permutation matrices is known as a *permutation representation*. A regular representation of a group G of degree n provides an embedding of G into the symmetric group S_n - this is asserted in the Cayley's Theorem

The book by Gordon and Martin [3] provides more details and more concrete examples of group representations in a leisurely introductory manner.

2.2 Reducibility and the Maschke's Theorem

We always gain a better understanding of a mathematical object by studying its sub-objects. In the theory of modules, we gain a better understanding of a module V if we can break it up into smaller modules called the submodules. Similarly in matrix theory, we find it easy to handle a matrix that have been transformed into its *reduced form*, i.e. either triangular or diagonal or the variations of the two in terms of block matrices. These definitions are useful in the formal statement and proof of Maschke's theorem

Definition 2.2.1 Let V be a G-module. A subspace W of V is said to be a *G-submodule* if it is invariant under the action of G i.e.

$$w \in W \Rightarrow gw \in W \quad \text{for all } g \in G$$

Definition 2.2.2 A non-zero G-module V said to be *reducible* if it has a non trivial G-submodule, otherwise V is *irreducible*. On the other hand, a matrix representation X corresponding to a G-module V is reducible if V is reducible, otherwise X is irreducible

Definition 2.2.3 Let X be a matrix, then X is a *direct sum of matrices* A and B written $X = A \oplus B$, if X has the block diagonal form

$$X = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

Theorem 2.2.1 (Maschke's Theorem) Let G be a finite group and $\mathbb{F}$ be a field such that char $\mathbb{F} \nmid |G|$ and let V be a non-zero $\mathbb{F}G$-module. Then

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)}$$

where each $W^{(i)}$ is an irreducible $\mathbb{F}G$-submodule of V. Equivalently, Maschke's theorem states that every representation T of a finite group G is equivalent with a direct sum of irreducible representations.

The proof of this theorem has been provided in the book by Lederman[4] and also a similar one has been given in [5] and is not very complicated. The proof provided by Sagan in [6] is very simple and straight forward. The following is a concrete example based on the proof provided in [6].

Example 2.2.1 Let $G = S_3$ - the group of all permutations of the objects in the set $S = \{1, 2, 3\}$, we can construct $\mathbb{C}\mathbf{S} = \mathbb{C}\{1, 2, 3\}$ as an S_3-module basing on the construction of the regular G-module as in example 2.1.2. The subspace W generated by the vector $(1 + 2 + 3)$ is an S_3-submodule since for any $\pi \in S_3$ and $w = c(1 + 2 + 3) \in W \quad (c \in \mathbb{C})$ we have

$$\begin{aligned}\pi w &= \pi c(1 + 2 + 3)\\ &= c(\pi(1) + \pi(2) + \pi(3))\\ &= c(1 + 2 + 3) \text{ up to re-arrangement}\\ &\in W\end{aligned}$$

W is of dimension 1 and hence is irreducible. Thus we would wish to express $\mathbb{C}\mathbf{S}$ as $\mathbb{C}\mathbf{S} = W \oplus W_1$ where W_1 is also an S_3-submodule. W is a subspace of $\mathbb{C}\mathbf{S}$, hence from the theory of vector spaces we know that $\mathbb{C}\mathbf{S} = W \oplus W^{\perp}$. Therefore, it suffices for us to find $W^{\perp}$ and to show that it is an S_3-submodule. On the other hand we can find $W^{\perp}$ by use of a S_3-*invariant inner product* $\langle \, , \rangle$ defined on $\mathbb{C}\mathbf{S}$ i.e. $\langle \pi v_1, \pi v_2 \rangle = \langle v_1, v_2 \rangle$ for all $\pi \in S_3$ and for all $v_1, v_2 \in \mathbb{C}\mathbf{S}$. We can define such an inner product between two vectors $v_1 = c_1 1 + c_2 2 + c_3 3$, $v_2 = d_1 1 + d_2 2 + d_3 3$ as

$$\langle v_1, v_2 \rangle = c_1 \bar{d}_1 + c_2 \bar{d}_2 + c_3 \bar{d}_3$$

where the bar denotes complex conjugation. This inner product is S_3-invariant, consequently we can give $W^{\perp}$ as

$$\begin{aligned}W^{\perp} = \mathbb{C}\{1 + 2 + 3\}^{\perp} &= \{v = a1 + b2 + c3 \;\; \langle v, 1 + 2 + 4 \rangle = 0\}\\ &= \{v = a1 + b2 + c3 : a + b - c = 0\}\end{aligned}$$

That is $W^\perp$ consist of all those vectors in $\mathbb{C}S$ whose co-ordinates with respect to the basis elements add up to zero, e.g. $1-2, 3-2, 3-1, \ldots$ Two linearly independent vectors will span $W^\perp$. For this example lets consider the vectors $1-2, 3-2$. Putting together the bases $\mathcal{B}_W = \{1+2+3\}, \mathcal{B}_{W^\perp} = \{1-2, 3-2\}$ for W and $W^\perp$ respectively we form a basis $\mathcal{B} = \{1+2+3, 1-2, 3-2\}$ for $\mathbb{C}S$. With this basis, every permutation $\pi \in S_3$ has a matrix representation which is the matrix of endomorphism $[\pi]_\mathcal{B}$. We get the following matrices

$$[\epsilon]_\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [(1,2)]_\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$[(1,3)]_\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad [(2,3)]_\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$[(1,2,3)]_\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad [(1,3,2)]_\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Clearly all the matrices in this representation are in the form

$$X(\pi) = \left(\begin{array}{c|c} A(g) & 0 \\ \hline 0 & B(g) \end{array}\right) \tag{2.2.1}$$

Now it remains to show that $W^\perp$ is indeed an S_3-submodule. We do this by considering a general case. If U is a G-submodule then $U^\perp$ is also a G-submodule as well. to show this take $w \in U^\perp$ and $g \in G$ then for all $u \in U$

$$\begin{aligned} \langle gw, u\rangle &= \langle g^{-1}gw, g^{-1}u\rangle \text{ (G-invariance of the inner product)} \\ &= \langle w, g^{-1}u\rangle \text{ (properties of groups)} \\ &= 0 \end{aligned}$$

This means that $gw \in U^{\perp}$ i.e. $U^{\perp}$ is a G-submodule. Finally, $W^{\perp}$ can also be shown to be irreducible and hence $\mathbb{C}\mathbf{S} = W \oplus W^{\perp}$ is an irreducible decomposition of the S_3-module just as stated in the Maschke's theorem

2.3 Group Characters

Let X be a representation of a group G. For each $g \in G$ define a mapping $\chi : G \rightarrow \mathbb{C}$ by $\chi(g) = \text{tr } X(g)$, then χ is called the character of the representation X. Characters of representations have many remarkable properties, and they are the fundamental tools for performing calculations in representation theory. Moreover, basic problems such as deciding whether or not a representation is irreducible, can be resolved by doing some easy arithmetic with the character of the representation.

Equivalent representations as well as isomorphic G-modules can be shown to have same character. Furthermore, if $g, h \in G$ are conjugate elements then $\chi(g) = \chi(h)$ for all characters χ of G. If V is a G-module, then the dimension of V is called the *degree* of χ, also if V is an irreducible G-module then the character χ is said to be *irreducible* otherwise it is a *reducible character*.

The regular character of G is the character corresponding to the reqular G-module and is denoted χ_{reg}

Proposition 2.3.1 Let V be a G-module, and suppose that

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_r$$

a direct sum of irreducible G-modules U_i then the character of V is equal to the sum of the characters of the G-modules $U_1, \ldots, U_r$

2.3.1 Permutation Characters

Let V be an n dimensional vector space with basis $\{v_1, v_2, \dots v_n\}$. For each $\pi \in S_n$, define the multiplication πv_i by

$$\pi v_i = v_{\pi(i)}$$

Extending this definition to be linear on V shows that V is an S_n-module, called the *permutation module*. Now define the set

$$\text{fix}(\pi) = \{i : 1 \leqslant i \leqslant n \text{ and } \pi(i) = i\}$$

The corresponding character χ also called the *permutation character* is therefore given by

$$\chi(\pi) = |\text{fix}(\pi)|$$

2.3.2 Inner Product of Characters

We can think of a character χ of a group $G = \{g_1, g_2, \dots, g_n\}$ as a row vector of complex numbers

$$\chi = (\chi(g_1), \chi(g_2), \dots, \chi(g_n))$$

Definition 2.3.1 Let χ and ψ be two characters of a group G. Then the inner product of χ and ψ is

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

where the bar denotes complex conjugation. It is shown in [6](section 1.9) that $\overline{\psi(g)} = \psi(g^{-1})$ giving us an alternative definition of the inner product as

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1})$$

Taking inner product of characters is a simple method of determining whether a representation is irreducible. In fact it has been shown in [3](Theorem 14.12) that if χ and ψ are inequivalent irreducible characters then

$$\langle \chi, \chi \rangle = 1 \text{ and}$$
$$\langle \chi, \psi \rangle = 0$$

This is called the Schur's orthogonality relation.

2.4 Restricted and Induced Representations

Let H be a subgroup of a group G and X be a matrix representation of G. The restriction of X to H, $X\downarrow_H^G$ is given by

$$X\downarrow_H^G = X(h) \text{ for all } h \in H$$

This mapping is actually a representation of the subgroup H and can be verified with no complications. If X has a character χ, then the character of $X\downarrow_H^G$ is denoted by $\chi\downarrow_H^G$. It is therefore easy to obtain a representation of a subgroup H of a group G from a representation of the group itself. The other way round may not be that trivial - that is given a representation of a subgroup H how can we obtain a representation of the whole group G? Frobenius succeeded in constructing a representation of a group from that of an arbitrary subgroup, from his procedure, as depicted in the following example

Example 2.4.1 Let $G = S_3$, $H = \langle (123) \rangle$. Let Y be the one dimensional representation of H given by

$$(123) \longrightarrow e^{\frac{2\pi i}{3}}$$

Our aim is to obtain a representation X of the entire group S_3 We start by decomposing S_3 into the disjoint left cosets of H. (Lets consider the transversal $(1), (12)$ i.e. $\mathcal{H} = \{(1)H, (12)H\}$) and for any $\pi \in S_3$ we define the mapping

$$X(\pi) = Y\uparrow_H^G(\pi) = \begin{pmatrix} Y((1)^{-1}\pi(1)) & Y((1)^{-1}\pi(12)) \\ Y((12)^{-1}\pi(1)) & Y((12)^{-1}\pi(12)) \end{pmatrix} \quad (2.4.1)$$

where $Y(\pi) = 0$ if $\pi \notin H$. For instance if $\pi = (23)$ then we have

$$(1)^{-1}(23)(1) = (23), \quad (1)^{-1}(23)(12) = (132).$$
$$(12)^{-1}(23)(1) = (123), \quad (12)^{-1}(23)(12) = (13)$$

therefore.

$$X((23)) = Y\uparrow_H^G((23)) = \begin{pmatrix} Y((23)) & Y((123)) \\ Y((132)) & Y((13)) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & e^{\frac{4\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & 0 \end{pmatrix}$$

Computation for the rest of matrices for the remaining elements is done in a similar manner and it is not hard to verify that the mapping is actually a representation. What it remains to show now is that the mapping defined in 2.4.1 is a representation irrespective on the group and the subgroup in consideration. The following definition is attributed to Frobenius:

Definition 2.4.1 Let H be a subgroup of G and $t_1, t_2, \ldots, t_k$ be a transversal for the left cosets of H. If Y is a representation of H. then the corresponding *induced representation* $Y\uparrow_H^G$ assigns to each $g \in G$ the block matrix

$$Y\uparrow_H^G(g) = \begin{pmatrix} Y(t_1^{-1}gt_1) & Y(t_1^{-1}gt_2) & \cdots & Y(t_1^{-1}gt_k) \\ Y(t_2^{-1}gt_1) & Y(t_2^{-1}gt_2) & \cdots & Y(t_2^{-1}gt_k) \\ \vdots & \vdots & & \\ Y(t_k^{-1}gt_1) & Y(t_k^{-1}gt_2) & \cdots & Y(t_k^{-1}gt_k) \end{pmatrix}$$

where $Y(g) = 0$ if $g \notin H$.

Frobenius actually discovered that $Y\uparrow_H^G(g)$ is indeed a representation, that is

$$Y\uparrow_H^G(g)Y\uparrow_H^G(h) = Y\uparrow_H^G(gh) \text{ for all } g, h \in G \qquad (2.4.2)$$

To show that equation 2.4.2 above is indeed a representation, we need to prove the following:

1. $Y\uparrow_H^G$ is always a block permutation matrix

2. $Y\uparrow_H^G(\epsilon)$ is the identity matrix

3. $\sum_{r=1}^{k} Y(t_i^{-1}gt_r)Y(t_r^{-1}ht_j) = Y(t_i^{-1}ght_j)$

(1). For a fixed i we show that $Y(t_i^{-1}gt_j)$ is non-zero for exactly one j. If we assume the contrary, say $Y(t_i^{-1}gt_j)$ and $Y(t_i^{-1}gt_k)$, $j \neq k$ are non-zero block matrices, then from the definition $t_i^{-1}gt_j \in H$ and $t_i^{-1}gt_k \in H \Rightarrow \exists x, y \in H$ such that $t_i^{-1}gt_jx = t_i^{-1}gt_ky$. By cancellation laws $t_jx = t_ky$ this contradicts the fact that t_j, t_k are representatives of two disjoint left cosets of H. Similar arguments show that for a fixed j, $Y(t_i^{-1}gt_j)$ is non-zero for exactly one i. The block permutation form ensures that the rows and columns of the resultant matrix are pairwise linearly independent respectively. This implies that the resultant matrix is invertible.

(2). From the arguments above $t_i^{-1}\epsilon t_j = t_i^{-1}t_j \in H \Leftrightarrow i = j \Rightarrow$ only the block matrices in the main diagonal are non-zero identity matrices.

(3). If we compare the (i,j)th block on both sides of the equation 2.4.2, it implies that

$$\sum_{r=1}^{k} Y(t_i^{-1}gt_r)Y(t_r^{-1}ht_j) = Y(t_i^{-1}ght_j) \qquad (2.4.3)$$

We have two cases arising, either $t_i^{-1}ght_j \in H$ or $t_i^{-1}ght_j \notin H$.

We note that $t_i^{-1}ght_j = t_i^{-1}gt_rt_r^{-1}ht_j$, therefore if $t_i^{-1}ght_j \notin H \Rightarrow Y(t_i^{-1}ght_j) =$

0 then either $t_i^{-1}gt_r \notin H$ or $t_r^{-1}ht_j \notin H$ for all $r \Rightarrow$ either $Y(t_i^{-1}gt_r) = 0$ or $Y(t_r^{-1}ht_j) = 0$ for all $r \Rightarrow$ forcing their product to be the zero block matrix. On the other case if $t_i^{-1}ght_j \in H$. let m be the unique index such that $t_i^{-1}gt_m \in H$, but then $t_m^{-1}ht_j = (t_i^{-1}gt_m)^{-1}t_i^{-1}ght_j \in H$,that is $t_i^{-1}gt_m.t_m^{-1}ht_j \in H$. Since Y is a representation defined on H then

$$Y(t_i^{-1}ght_j) = Y(t_i^{-1}gt_mt_m^{-1}ht_j) = Y(t_i^{-1}gt_m)Y(t_m^{-1}ht_j) \tag{2.4.4}$$

This proves 2.4.3 above and consequently part (3) above and consequently proves that 2.4.2 is actually a representation. We also note that induced and restricted representations do not preserve irreducibility and further more induced representations depend only on the subgroup chosen and not the transversal. This is asserted in the following proposition (without Proof)

Proposition 2.4.1 Consider $H \leq G$ and a matrix representation Y of H. Let $\{t_1, \ldots, t_l\}$ and $\{s_1, \ldots, s_l\}$ be two transversals for H giving rise to representations matrices X and Z. respectively, for $Y \uparrow_H^G$. Then X and Z are equivalent.

The proof of this proposition can be found in the book by Sagan [6]

Corollary 2.4.1 Let G be a group and K. H be subgroups of G such that $K \leq H \leq G$ and Y a matrix representation of K. Then

$$(Y \uparrow_K^H) \uparrow_H^G \cong Y \uparrow_K^G \tag{2.4.5}$$

Proof: Let $T_1 = \{h_1, h_2, \ldots, h_m\}$ be a transversal for the left cosets of K in H. $T_2 = \{g_1, g_2, \ldots, g_n\}$ be a transversal for the left cosets of H in G and $T_3 = \{a_1, a_2, \ldots, a_p\}$ be a transversal for the left cosets of K in G. First we note that

$$\left.\begin{aligned} |H| &= m|K| \\ |G| &= n|H| = nm|K| \\ |G| &= p|K| \end{aligned}\right\} \Rightarrow p = nm$$

Let $T = \{g_i h_j | g_i \in T_2, h_j \in T_1, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$

Claim: The set T is a transversal for the left cosets of K in G. To show this, suppose that there exist $g_i, g_l \in T_2$ and $h_j, h_d \in T_1$ such that

$$g_i h_j K \cap g_l h_d K \neq \phi \tag{2.4.6}$$

We have the following cases: either $i = l$ or $i \neq l$. If $i \neq l$ then since $h_j K \subset H$ and $h_d K \subset H$ and g_i, g_l are representatives of disjoint left cosets of H, then $g_i h_j K \cap g_l h_d K = \phi$ contradicting 2.4.6 above. On the other hand if $i = l$ the cancellation laws imply that 2.4.6 above can be written as

$$h_j K \cap h_d K \neq \phi \tag{2.4.7}$$

Similarly h_j, h_d are representatives of their respective left cosets of K in H, hence if $j \neq d$ then the above cannot be true. This means that 2.4.6 holds if and only if $g_i = g_l$ and $h_j = h_d$, consequently the left cosets $g_i h_j K$, $(g_i \in T_2, h_j \in T_1)$ are mutually disjoint each with cardinality $|K|$ and therefore they are in a $1-1$ correspondence with the left cosets $a_r K$ of K in G. This proves our claim.

Now going back to 2.4.5 above, we have

$$Y \uparrow_K^H (x) = Y(h_j^{-1} x h_d) \text{ where } j, d \in \{1, 2, \ldots, m\}$$

$$\begin{aligned}
(Y \uparrow_K^H) \uparrow_H^G (x) &= Y \uparrow_K^H (g_i^{-1} x g_l) \text{ where } i, l \in \{1, 2, \ldots, n\} \\
&= Y(h_j^{-1}(g_i^{-1} x g_l) h_d) \\
&= Y((g_i h_j)^{-1} x (g_l h_d))
\end{aligned} \tag{2.4.8}$$

$$Y \uparrow_K^G (x) = Y(a_r^{-1} x a_w) \text{ where } r, w \in \{1, 2, \ldots, p\} \tag{2.4.9}$$

Finally as a corollary of proposition 2.4.1 above, 2.4.8 and 2.4.9 imply that the resultant representations are equivalent.

Chapter 3

Construction of Ordinary Irreducible Modules of S_n

It has been proved in Sagan [6] that the number of all irreducible representations of S_n is equal to the conjugacy classes of S_n. On the other hand we have already seen that the number of possible partitions of n is equal to the number of conjugacy classes of S_n. This means that for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ there correspond an irreducible S_n-module. Our main aim in this section is to construct such an irreducible module.

Definition 3.0.2 For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n(also written $\lambda \vdash n$), we can find a corresponding subgroup S_λ by taking an inner product $S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_k+1,n-\lambda_k-2,\ldots,n\}}$. This subgroup is called the *Young's subgroup* of S_n.

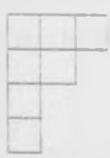
As an illustration, if $n = 3$ and we take a partition of 3 say $\lambda = (2, 1)$, then the corresponding Young Subgroup $S_{(2,1)}$ is given as

$$\begin{aligned} S_{(2,1)} &= S_{\{1,2\}} \times S_{\{3\}} \\ &= \{(1)(2), (1,2)\} \times \{(3)\} \\ &= \{(1)(2)(3), (1,2)(3)\} \\ &= \{1, (1,2)\} \end{aligned}$$

Taking the Young's subgroup S_λ corresponding to a partition λ and inducing the trivial representation on S_λ up to S_n, we get a representation $1 \uparrow_{S_\lambda}^{S_n}$ and consequently we get its corresponding module M^λ, which may not be irreducible in general. However, the set $\{\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \dots\}$ of partitions of n can be ordered such that the first module $M^{\lambda^{(1)}}$ is irreducible and we can call it $S^{\lambda^{(1)}}$. Next, $M^{\lambda^{(2)}}$ will contain only copies of $S^{\lambda^{(1)}}$ and a copy of a new irreducible $S^{\lambda^{(2)}}$. In general $M^{\lambda^{(i)}}$ will decompose into some $S^{\lambda^{(j)}}$ for $j < i$ and a new unique irreducible $S^{\lambda^{(i)}}$ called the ith Specht module.

3.1 Young's Diagrams and Tableaux

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n. The diagram $[\lambda]$ of λ is an ordered set of boxes with λ_i boxes in row i. The boxes also called *nodes* may be labeled (i, j) and thus the young diagram may be defined as $[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} | j \leq \lambda_i\}$. Going back to section 1.2 above we used $\lambda = (3, 2, 1^2)$ as an example of partition. Its corresponding Young's diagram is



Definition 3.1.1 A **Young's tableau** t of shape λ or just λ-tableau is an array of integers obtained by placing the integers from 1 through n into the nodes of the diagram of λ in a bijective manner.

Example 3.1.1 For example if $n = 3$ and $\lambda = (2, 1)$, some of the possible tableaux of shape λ are

$$t_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad t_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad t_3 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

For clarity, the grid lines in the diagram may be dropped

Definition 3.1.2 Two tableaux t_1, t_2 of shape λ are *row equivalent* $t_1 \sim t_2$, if corresponding rows of the two tableaux contains the same elements. This is an equivalence relation and consequently, the entire set of λ-tableaux can be decomposed into distinct sets of disjoint equivalence classes.

Definition 3.1.3 A *tabloid of shape* λ or λ-*tabloid* is then $\{t\} = \{t_1 | t_1 \sim t\}$ That is the tabloid $\{t\}$ is the equivalence class of t.

Thus if

$$t = \begin{array}{cc} 2 & 3 \\ 1 & \end{array}$$

then

$$\{t\} = \left\{ \begin{array}{cc} 2 & 3 \\ 1 & \end{array} \quad \begin{array}{cc} 3 & 2 \\ 1 & \end{array} \right\} = \begin{array}{cc} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}$$

The lines between the rows of the array is to show that it is a tabloid. It is easy to see that the number of λ-tabloids is just $\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_k!}$

S_n can act on a tableau of shape λ in the following manner. for each $\pi \in S_n$ let

$$\pi t = (\pi(t_{i,j}))$$

This induces an action on tabloids by letting

$$\pi\{t\} = \{\pi t\}$$

This action is well defined. i.e. if t and s are row equivalent tableaux of shape λ then the entries in row i of tableau t and s are in a $1-1$ correspondence. consequently if $t_{(i,j)} = s_{(i,r)}$ then $\pi(t_{(i,j)}) = \pi(s_{(i,r)})$. This means that the action induced on the set of all tabloids of shape λ is independent of the choice of tableau t representing $\{t\}$. This action gives rise to an S_n-module.

Definition 3.1.4 Suppose $\lambda \vdash n$. Let

$$M^\lambda = \mathbb{C}\{\{\mathrm{t}_1\}, \{\mathrm{t}_2\}, \ldots, \{\mathrm{t}_r\}\}$$

where $\{t_1\}$ $\{t_2\}, \ldots, \{t_r\}$ is a complete list of λ-tabloids. Then M^λ is called the *permutation module corresponding to* λ

Example 3.1.2 Let $\lambda = (2,1) \vdash 3$. M^λ is the vector space over the complex field $\mathbb{C}$ spanned by the set of tabloids below

$$\{\mathrm{t}_1\} = \begin{array}{c} \hline 2\ \ 3 \\ \hline 1 \\ \hline \end{array}, \qquad \{\mathrm{t}_2\} = \begin{array}{c} \hline 1\ \ 3 \\ \hline 2 \\ \hline \end{array}, \qquad \{\mathrm{t}_2\} = \begin{array}{c} \hline 1\ \ 2 \\ \hline 3 \\ \hline \end{array}$$

After constructing the S_n modules M^λ for each partition $\lambda \vdash n$ our next step is to put these modules in some order that will have the properties earlier described.

3.2 Ordering

The set of all partitions of n is partially ordered as in the following defintion

Definition 3.2.1 If λ and μ are partitions of n then we say that λ *dominates* μ written $\lambda \trianglerighteq \mu$ provided that for all j

$$\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i$$

This is a partial order since there exist partitions that cannot be compared in this manner e.g. $\lambda^{(1)} = (3, 1, 1, 1)$ and $\lambda^{(2)} = (2, 2, 2)$ cannot be compared. Another ordering that can be defined on the set of all partitions of n is the *dictionary order(lexicographical order)*

Definition 3.2.2 If λ and μ are partitions of n then we write $\lambda > \mu$ if and only if then least j for which $\lambda_j \neq \mu_j$ satisfies the relation $\lambda_j > \mu_j$

Lemma 3.2.1 (Dominance Lemma for Partitions) Let λ and μ be partitions of n and suppose that t_1 is a λ-tableau and t_2 is a μ-tableau. Suppose that for every i the numbers from the ith row of t_2 belong to different columns of t_1 then $\lambda \trianglerighteq \mu$

Proof. Suppose that we place μ_1 numbers from the first row of t_2 in t_1 such that no two numbers are in the same column. Then t_2 must have at least μ_1 columns that is $\lambda_1 \geqslant \mu_1$. Next insert the μ_2 numbers from the second row of t_2 in different columns. For this to be possible, we require $\lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2$. Continuing this way, we have $\lambda \trianglerighteq \mu$.

The total order $>$ contains the partial order $\triangleright$, in the sense that if $\lambda \triangleright \mu$ then $\lambda > \mu$, but the reverse implication is false in general

If we list M^λ in a *dual* lexicographical order then the first module $M^{(n)}$ is one dimensional and will be irreducible to start with. This order will have the property described in the introductory paragraphs of this chapter. For example if $n = 6$ then:

$$(6) > (5, 1) > (4, 2) > (4, 1^2) > (3^2) > (3, 2, 1) > (3, 1^3) > (2^3) > (2^2, 1^2) > (2, 1^4) > (1^6)$$

$M^{(6)} = S^{(6)}$ will be an irreducible module of S_6, $M^{(5,1)}$ will contain one copy of a new irreducible $S^{(5,1)}$ and copies of $S^{(6)}$, just as in the property earlier described

3.3 Constructing the Specht Module Corresponding to a Partition λ

Definition 3.3.1 Let t be a λ-tableau, then its *column-stabilizer* C_t is the Young's subgroup of S_n keeping the columns of t fixed setwise that is

$$C_t = \{\pi \in S_n : i \text{ and } \pi(i) \text{ belong to the same column of } t\}$$

If t has columns $C_1, C_2, \ldots, C_k$ then its column stabilizer is

$$C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$$

Definition 3.3.2 If t is a λ-tableau the procedure for constructing S^λ starts by setting the *signed column sum* $\kappa_t \in \mathbb{C}S_n$ which is obtained by summing the elements in the column stabilizer of t and attaching a signature to each permutation. i.e.

$$\kappa_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi$$

As t varies over all λ-tableaux in the tabloid $\{t\}$ we form the product $e_t = \kappa_t\{\mathrm{t}\}$ called a *polytabloid* associated with the the tableau t

Example 3.3.1 If $t = \begin{matrix} 3 & 2 & 4 \\ 1 & 5 & \end{matrix}$ then $C_t = \{1, (1,3), (2,5), (1,3)(2,5)\}$ and consequently $\kappa_t = 1 - (1,3) - (2,5) + (1,3)(2,5)$ where 1 denotes the identity element. We then form a polytabloid e_t

$$e_t = \frac{\overline{\begin{matrix} 3 & 2 & 4 \end{matrix}}}{\underline{\begin{matrix} 1 & 5 \end{matrix}}} - \frac{\overline{\begin{matrix} 1 & 2 & 4 \end{matrix}}}{\underline{\begin{matrix} 3 & 5 \end{matrix}}} - \frac{\overline{\begin{matrix} 3 & 5 & 4 \end{matrix}}}{\underline{\begin{matrix} 1 & 2 \end{matrix}}} + \frac{\overline{\begin{matrix} 1 & 5 & 4 \end{matrix}}}{\underline{\begin{matrix} 3 & 2 \end{matrix}}}$$

Its a logical question to ask about the relation between the polytabloids formed by applying the process above on to different tabloids corresponding to a partition λ. We know that if $x_1, x_2 \in S_n$ have the same cycle-type then, there exists a permutation π such that $\pi x_1 \pi^{-1} = x_2$. Consequently, if t_1 and t_2 are two different λ-tableaux in different λ-tabloids then there exists a permutation $\pi \in S_n$ such that $t_1 = \pi t_2 \pi^{-1}$ The following lemma clarify this relation.

Lemma 3.3.1 Let t be a tableau and π be a permutation. Then

1. $C_{\pi t} = \pi C_t \pi^{-1}$
2. $\kappa_{\pi t} = \pi \kappa_t \pi^{-1}$
3. $\mathbf{e}_{\pi t} = \pi \mathbf{e}_t$

Proof. Let $\lambda = (\lambda_1 . \lambda_2 . \dots . \lambda_k)$ be a partition of n and t be a tableau of shape λ. A general tableau of t is given by

$$t = \begin{matrix} t_{1,1} & t_{1,2} & \cdot & t_{1,\lambda_1} \\ t_{2,1} & t_{2,2} & \cdots & t_{2,\lambda_2} \\ \vdots & \vdots & & \\ t_{k,1} & t_{k,2} & \cdots & t_{k,\lambda_k} \end{matrix}$$

The tableau t has columns $C_1, C_2 \dots, C_{\lambda_1}$ and hence its column stabilizer C_t will be $C_t = S_{C_1} \times S_{C_2} \times \cdot \quad \times S_{C_\lambda}$. A typical element $\sigma \in C_t$ will be given as a disjoint product of cycles of the form $(t_{i,j}, t_{i+1,j}, \dots t_{i+r,j})$. On the other hand. if a permutation π acts on the tableau t we get a tableau πt whose column stabilizer $C_{\pi t}$ will be given by $C_{\pi t} = S_{\pi C_1} \times S_{\pi C_2} \times \cdots \times S_{\pi C_{\lambda_1}}$. Consequently a typical element $\tau \in C_{\pi t}$ will be given by a product of disjoint cycles of the form $(\pi t_{i,j}, \pi t_{i+1,j}, \dots, \pi t_{i+r,j})$. Thus to prove part 1 of lemma 3.3 1 above we need to show

$$(\pi t_{i,j} . \pi t_{i+1,j} . \dots , \pi t_{i+r,j}) = \pi (t_{i,j} . t_{i+1,j}, \dots , t_{i+r,j}) \pi^{-1} \tag{3.3.1}$$

By proposition 1.3.1 of section 1.3 above, the equation 3.3.1 holds true. The proof of part 2 proceeds similarly. Finally to prove part 3 we have

$$\mathbf{e}_{\pi\mathbf{t}} = \kappa_{\pi t}\{\pi\mathbf{t}\} = \pi\kappa_t\pi^{-1}\{\pi\mathbf{t}\} = \pi\kappa_t\{\mathbf{t}\} = \pi\mathbf{e}_t$$

Definition 3.3.3 For any partition λ, the corresponding *Specht module* S^λ is the submodule of M^λ spanned by the polytabloids $\mathbf{e}_t$, where t is the shape of λ

Example 3.3.2 We will use $n = 3$ to illustrate the entire process of constructing all the Specht modules and there after, a generalization for a general n will be given. The choice of $n = 3$ is entirely to keep our calculations as simpler as possible. $n = 3$ has three partitions namely $\lambda^{(1)} = (3) > \lambda^{(2)} = (2,1) > \lambda^{(3)} = (1,1,1)$ listed in the dual lexicographical order.

For $\lambda^{(1)} = (3)$ we have one unique tabloid

$$\{t\} = \begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline \end{array}$$

Thus $M^{(3)} = \mathbb{C}\{\{\mathbf{t}\}\}$ is one dimensional and thus is irreducible. Note also that as t varies over all tableaux of shape $\lambda^{(1)} = (3)$ our procedure as described produces polytabloids which are all equal and hence the Specht module $S^{(3)}$ being the span of all the resultant polytabloids will just reduce to the span of a single polytabloid. It is not difficult to show that the same is true for the case $\lambda = (n)$

For $\lambda^{(2)} = (2,1)$ we have three distinct tabloids

$$\{t_1\} = \begin{array}{cc} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}, \qquad \{t_2\} = \begin{array}{cc} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}, \qquad \{t_3\} = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

Consequently $M^{(2,1)} = \mathbb{C}\{\{\mathbf{t}_1\},\{\mathbf{t}_2\},\{\mathbf{t}_3\}\}$ is 3-dimensional and is also reducible. We therefore aim to extract a submodule $S^{(2,1)}$ which is irreducible. We start by considering a tabloid $\{t_i\}$ and for each tableau say $t_{i,j}$ in this tabloid we form the Young's subgroup $C_{i,j}$ stabilizing its columns. We then form the signed column sum $\kappa_{i,j}$ and finally we form a polytabloid $\mathbf{e}_{ij}$. The second Specht module $S^{(2,1)}$ is therefore the span of the resultant polytabloids $e_{i,j}$. To make things more concrete, lets consider the tabloid $\{t_2\}$. We have two tableaux in this tabloid

$$t_{2,1} = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \qquad t_{2,2} = \begin{array}{cc} 3 & 1 \\ 2 & \end{array}$$

The tableau $t_{2,1}$ has its corresponding Young's subgroup stabilizing the columns is $C_{2,1} = \{1, (1,2)\}$. Consequently, the signed column sum $\kappa_{2,1} = 1 - (1,2)$ and finally we have the polytabloid

$$\mathrm{e}_{2,1} = \begin{array}{cc} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} - \begin{array}{cc} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array}$$

A similar procedure on the tableau $t_{2,2}$ yields the polytabloid

$$\mathrm{e}_{2,2} = \begin{array}{cc} \hline 3 & 1 \\ \hline 2 & \\ \hline \end{array} - \begin{array}{cc} \hline 2 & 1 \\ \hline 3 & \\ \hline \end{array}$$

Note that $\mathrm{e}_{2,1}$ and $\mathrm{e}_{2,2}$ are linearly independent in the space $M^{(2,1)}$. The Specht module $S^{(2,1)}$ is thus the span of the two polytabloids. Note also that we could have started with any tabloid t_i and still end up with the same Specht Module $S^{(2,1)}$, lemma 3.3.1 above proves this case.

Finally, for $\lambda^{(3)} = (1,1,1) = (1^3)$ we have six distinct tabloids. Just as above, we will consider only one tabloid say

$$\{t\} = \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

There is only one tableau t in this tabloid and the corresponding Young's subgroup stabilizing the columns of t is the symmetric group S_3. This yields the signed column sum

$\kappa_t = 1 + (1,2,3) + (1,3,2) - (1,2) - (1,3) - (2,3)$ and consequently we get a single polytabloid

$$\mathbf{e}_t = \begin{array}{c}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} + \begin{array}{c}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array} + \begin{array}{c}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline\end{array} - \begin{array}{c}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline\end{array} - \begin{array}{c}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} - \begin{array}{c}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}$$

$S^{(1^3)}$ is the span of e_t, it is one dimensional and thus is irreducible. We have therefore succeeded in constructing all the irreducible modules of S_3. We note that if $\lambda = (1^n)$ and t a λ-tableau then

$$\kappa_t = \sum_{\sigma \in S_n} (\text{sgn } \sigma)\sigma$$

Consequently $\mathbf{e}_t$ is the signed sum of all $n!$ permutations regarded as tabloids. Now for any permutation π lemma 3.3.1 yields

$$\mathbf{e}_{\pi t} = \pi \mathbf{e}_t = \sum_{\sigma \in S_n} (\text{sgn } \sigma)\pi\sigma\{\mathbf{t}\}$$

Substituting $\tau = \pi\sigma$

$$\mathbf{e}_{\pi t} = \sum_{\tau \in S_n} (\text{sgn } \pi^{-1}\tau)\tau\{\mathbf{t}\} = (\text{sgn } \pi^{-1}) \sum_{\tau \in S_n} (\text{sgn } \tau)\tau\{\mathbf{t}\} = (\text{sgn } \pi)\mathbf{e}_t$$

because sgn π = sgn π^{-1}. Thus every polytabloid is a scalar multiple of $\mathbf{e}_t$ so

$$S^{(1^n)} = \mathbb{C}\{\mathbf{e}_t\}$$

with the action $\pi\mathbf{e}_t = (\text{sgn } \pi)\mathbf{e}_t$. This is the sign representation.

All along it has been assumed that the entire resultant set of Specht modules S^λ in the process constitute a full set of all irreducible modules of S_n and are indeed irreducible. This is not just an assumption. the submodule theorem proves the Specht modules are indeed irreducible and further that all the Specht modules constructed as in the above process constitute a full set of irreducible S_n modules.

3.4 The Submodule Theorem

Given a subset $H \subseteq S_n$ and vectors $\{t\}, \{s\} \in M^\lambda$ define the group algebra sum H^- and the S_n invariant inner product $\langle \cdot, \cdot \rangle$ by

$$H^- = \sum_{\pi \in H} \operatorname{sgn}(\pi)\pi$$

$$\langle \{t\}, \{s\} \rangle = \delta_{\{t\},\{s\}}$$

Lemma 3.4.1 (Sign Lemma) Let $H \subseteq S_n$ be a subgroup then

1. If $\pi \in H$ then

$$\pi H^- = H^- \pi = \operatorname{sgn}(\pi) H^-$$

2. For any $\mathbf{u}, \mathbf{v} \in M^\lambda$

$$\langle H^- \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, H^- \mathbf{v} \rangle$$

3. If the transposition $(b, c) \in H$ then we can factor

$$H^- = k(1 - (b, c)), \quad k \in \mathbb{C}[S_n]$$

4. If t is a tableau with b, c in the same row of t and $(b, c) \in H$ then

$$H^- \{t\} = 0$$

The sign lemma has been proved in [6](Lemma 2.4.1) and help us to prove these corollaries

Corollary 3.4.1 Let t and s be tableaux of shape λ and μ respectively with $\lambda, \mu \vdash n$. If $\kappa_t\{s\} \neq 0$ then $\lambda \trianglerighteq \mu$ and if $\lambda = \mu$ then $\kappa_t\{s\} = \pm\{e_t\}$

Corollary 3.4.2 If $\mathbf{u} \in M^\mu$ and t is a tableau of shape μ then $\kappa_t \mathbf{u}$ is a scalar multiple of e_t.

We are now in a position to state and prove the main result in this section

Theorem 3.4.1 (Submodule Theorem) Let U be a submodule of M^{μ}, then

$$U \supseteq S^{\mu} \text{ or } U \subseteq S^{\mu-}$$

In particular if M^{μ} is a module over $\mathbb{C}$. then the S^{μ} are irreducible. The layman statement of this theorem is that the *Specht Module* S^{μ} does not contain any other submodule of the module M^{μ}

Proof. Consider $\mathrm{u} \in U$ and a μ-tableau t. By corollary 3.4.2 above $\kappa_t \mathrm{u} = f e_t$ for some scalar f. Two cases arise:

Case 1: Suppose that there exists a μ and a t with $f \neq 0$. Then since $\mathrm{u} \in U$ we have $f e_t = \kappa_t \mathrm{u} \in U$ (Since f is non-zero) and $S^{\mu} \subseteq U$(Since S^{μ} is cyclic)

Case 2: Suppose that we always have $\kappa_t \mathrm{u} = 0$. Consider any $\mathrm{u} \in U$ and any arbitrary μ-tableau t, we can apply part 2 of the sign lemma to obtain

$$\begin{aligned}
\langle \mathrm{u}, e_t \rangle &= \langle \mathrm{u}. \kappa_t \{t\} \rangle \\
&= \langle \kappa_t \mathrm{u}. \{t\} \rangle \\
&= \langle 0, \{t\} \rangle \\
&= 0
\end{aligned}$$

Since e_t span S^{μ} we have $\mu \in S^{\mu-}$. Finally it has been proved in [6](Theorem 2.4.6) that the S^{λ} for $\lambda \vdash n$ form a complete list of irreducible S_n-modules over the complex field.

In chapter 2 we defined representations while restricting ourselves to the field of complex numbers. This should not imply that the only representations studied are over the field of complex complex numbers. In fact a lot interesting studies in representation theory arise when we consider other fields different from the field of complex numbers such as field of finite prime characteristic p. The theorem we just proved - the irreducibility of the Specht modules, entirely depends on the field over which the module M^{λ} is defined. The reducibility of Specht modules over the fields of finite characteristic is discussed in the next chapter.

Chapter 4

Modular Representation of the Symmetric Group

4.1 Historical Perspective

In representation theory of finite groups, it is useful to know which ordinary irreducible representations remain irreducible modulo p. For the symmetric groups S_n this amounts to determining which Specht modules are irreducible over a field of characteristic p. The definition of S^λ is independent of the field $\mathbb{F}$ we are working over, thus we write $S^\lambda_{\mathbb{F}}$ when we wish to draw attention to the ground field $\mathbb{F}$ or $S^\lambda_{\mathbb{F}_p}$ to draw attention about the field and further stress the fact that $\mathbb{F}$ is of characteristic p. Much of the ground work in this area has been dealt with in [7]

Definition 4.1.1 (p-regular, p-singular, p-restricted) A partition $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ is p-regular if it does not have p equal parts otherwise it is p-singular. It is p-restricted if there is no i with $\lambda_i - \lambda_{i+1} \geqslant p$

Let S^λ be a submodule of M^λ corresponding to a partition λ of n. In [9], the author proved the following

1. For any submodule U of M^λ, either $U \supseteq S^\lambda$ or $U \subseteq S^{\lambda\perp}$ (See theorem 3.4.1

above)

2. $S^\lambda \cap S^{\lambda\perp} \subset S^\lambda \Leftrightarrow \lambda$ is p-regular. In this case, (by 1 above) $S^\lambda \cap S^{\lambda\perp}$ is the unique maximal submodule of S^λ

3. Write $D^\lambda = S^\lambda / S^\lambda \cap S^{\lambda\perp}$ Then $D^\lambda \cong D^\mu$ if and only if $\lambda = \mu$(λ and μ p-regular)

4. If λ is p-regular, just one composition factor of M^λ is isomorphic to D^λ. The other composition factors have the form D^μ with $\mu > \lambda$

5. If λ is p-singular all composition factors of M^λ have the form D^μ with $\mu \rhd \lambda$

Given a partition $\lambda \vdash n$ and a field $\mathbb{F}$, we construct a Specht module S^λ and if char $\mathbb{F} = 0$. then S^λ is irreducible. As λ runs over all partitions of n. the submodules S^λ constitute all the irreducible representations of S_n. Otherwise, if char $\mathbb{F} = p$. then there is a canonically defined submodule $V^\lambda \subsetneq S^\lambda$. and V^λ is proper if and only if λ is p-regular. Let $D^\lambda = S^\lambda / V^\lambda$, then each D^λ is simple and as λ runs over all of the p-regular partitions of n then D^λ runs over all of the irreducible $\mathbb{F}S_n$-modules

4.1.1 Cater's Conjecture for Specht Modules

From Section 4.1 we saw that the D^λ's constitute all the irreducible p-modular representations of S_n. We are however concerned with the question "Which ordinary irreducible representations of the symmetric group S_n remain irreducible modulo a prime p?". R. W. Carter conjectured an answer to this question

Definition 4.1.2 (Hook and Hook Length) If $v = (i, j)$ is a node in the diagram of λ then it has a *hook*

$$H_v = H_{i,j} = \{(i, j') : j' \geq j\} \cup \{(i', j) : i' \geq i\}$$

with the corresponding *hook length*

$$h(v) = h(i, j) = |H_{i,j}|$$

Definition 4.1.3 (Hook Graph) The hook graph of a diagram $[\lambda]$ is a tableau corresponding to λ obtained when the content of every node (i,j) is its hook length $h(i,j)$

Definition 4.1.4 (p-power diagram, Definition 3.3 [10]) The p-power diagram $[\lambda]^p$ is obtained from the hook graph for $[\lambda]$ by replacing each entry $h(i,j)$ by $max\{a|p^a \text{ divides } h(i,j)\}$

Conjecture 4.1.1 (Carter) If λ is p-regular then S^λ is irreducible if and only if no column of the p-power diagram contains two different numbers

James proved in [11], that the condition in Carter's conjecture is necessary for S^λ to be irreducible and went ahead to prove that the conjecture is correct when $p = 2$.

4.1.2 James-Mathas Conjecture on the Reducibility of Specht Modules

In the article [12], James and Mathas classified the irreducible Specht modules for the case $p = 2$ and thereby verified the conjecture by Cater (Conjecture 4.1.1) in this case. In his book [13], Mathas also conjectured a necessary and sufficient condition that must be satisfied by the diagram of a partition $[\lambda]$ in order for the corresponding Specht module to be reducible on the case where p is an odd prime.

Conjecture 4.1.2 (Conjecture 5.47 [13]) If p is an odd prime, then the Specht module $S^\lambda_{\mathbb{F}_p}$ is reducible if and only if the Young diagram $[\lambda]$ contains nodes $(i,j), (i,k)$ and (r,j) such that

$$v_p(h(i,j)) > 0$$

and

$$v_p(h(r,j)) \neq v_p(h(i,j)) \neq v_p(h(i,k))$$

Where v_p is the p-adic valuation function and $h(i,j)$ the hook length of the node (i,j).

Lyle proved a major part of the necessary part of conjecture 4.1.2 in the article [14] and, thereafter, Fayers [15] build on the work of Lyle to complete the proof of the necessary part, and almost immediately, completed the proof of conjecture 4.1.2 by proving the sufficiency of the condition in [16].

4.1.3 More on the History of Modular Representations

The study of modular representation theory was in some sense started by H. E. Dickson in 1902. Between 1935 and 1977 Brauer almost single-handedly constructed the body of what is now regarded as the classical modular representation theory. Brauer's main motivation in studying modular representations was to obtain number theoretic restrictions on the possible behavior of ordinary character tables, and thereby find restrictions upon the structure of finite groups.

It was really J. A. Green Who first systematically developed the study of modular representation theory from the point of view of examining the set of indecomposable modules. Green's results were an indispensable tool in the treatment by Thompson, and then more fully by Dade, of blocks with cyclic defect groups. Since then, many other people have become interested in the study of the modules for their own sake

Chapter 5

Summary

5.1 Irreducible $\mathbb{F}S_n$-Modules

The definition of S^λ is independent of the field $\mathbb{F}$ we are working over - thus for any field $\mathbb{F}$ we may construct a Specht module S^λ. Depending on the characteristic of the field $\mathbb{F}$ and also the nature of the partition λ, S^λ may be irreducible

As summarized in [17], when char $\mathbb{F} = 0$. the Specht module S^λ is irreducible, and as λ ranges over all partitions of n. the modules S^λ range over all non-isomorphic irreducible $\mathbb{F}S_n$−modules. However. if char $\mathbb{F} = p > 0$. then $S^\lambda_\mathbb{F}$ may be reducible. but when λ is p−regular, then $S^\lambda_\mathbb{F}$ has a uniquely defined quotient module, D^λ, which is irreducible. Consequently as λ ranges over all p−regular partitions of n. the D^λ range over all distinct irreducible $\mathbb{F}S_n$−modules We note that when $\mathbb{F}$ has characteristic $p > 0$, we may always choose a basis so that $S^\lambda_\mathbb{F}$ may be viewed as being over $\mathbb{F}_p$

In other words, given any field $\mathbb{F}$ of any characteristic then it is possible for us to construct the irreducible $\mathbb{F}S_n$−modules. Therefore if our sole objective is to obtain the irreducible $\mathbb{F}S_n$−modules. then we will have achieved our goal. However.

if we are further interested in knowing which irreducible $\mathbb{F}S_n$-modules when char $\mathbb{F} = 0$ remains irreducible when we shift our base field to a field of char $\mathbb{F} = p > 0$, then these $\mathbb{F}S_n$-modules are classified as follows:

5.2 p-Irreducible Specht Modules

The case $p = 2$ is different from the general case: the classification of 2-irreducible Specht modules was dealt with by James and Mathas in [12]. Their main result was as follows:

Theorem 5.2.1 Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition. Then the Specht module S^λ is irreducible in characteristic 2 if and only if

1. $\lambda_i - \lambda_{i+1} \equiv -1 \bmod 2^{l(\lambda_{i+1} - \lambda_{i+2})}$ for all $i \geqslant 1$; or,
2. $\lambda'_i - \lambda'_{i+1} \equiv -1 \bmod 2^{l(\lambda'_{i+1} - \lambda'_{i+2})}$ for all $i \geqslant 1$ where $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$ is a partition conjugate to λ; or,
3. $\lambda = (2, 2)$

When p is an odd prime, then James and Mathas conjectured(conjecture 4.1.2) necessary and sufficient conditions that must be satisfied by a partition λ such that the corresponding Specht Module S^λ is irreducible. The conjecture has been proved and this enables us to classify all p-irreducible Specht modules irrespective of the characteristic of the field $\mathbb{F}$ over which the Specht module is defined.

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